Problem 1

Let $X_1, \ldots, X_n$ be a random sample from $\text{Exp}(\lambda)$, where $\lambda > 0$; i.e., the pdf of $X_1$ is $f(x) = \lambda^{-1} \exp(-x/\lambda)$ for $x > 0$ and zero otherwise.

(i) Show that $E[X_1] = \lambda$ and $\text{var}[X_1] = \lambda^2$.

(ii) Find $\hat{\lambda}$, the mle of $\lambda$.

(iii) Show that $\hat{\lambda}^{-1} \xrightarrow{p} \lambda^{-1}$ as $n \to \infty$.

(iv) Find the asymptotic distribution of $n^{1/2}(\hat{\lambda} - \lambda)$ as $n \to \infty$.

(v) Use (iv) to find an approximate 95% confidence interval for $\lambda$.

Problem 2

Let $X_1, \ldots, X_n$ be a random sample from a population with pdf $f(x, \lambda) = 3\lambda x^{-3} \exp(-\frac{1}{x^3\lambda})$ for $x > 0$, $\lambda > 0$.

(i) Show that $\hat{\lambda} = \sum_{i=1}^{n} X_i^{-3}/n$ is the mle of $\lambda$.

(ii) Show that $X_i^{-3} \overset{d}{\sim} \text{Exponential}(\lambda)$.

(iii) Use (ii) to calculate an approximately $(1 - \alpha)100\%$ confidence interval for $\lambda$.

Problem 3

Suppose that every person in a population belongs to either group $H$ or group $T$ and we want to estimate the proportion belonging to group $H$. For a concrete illustration, assume that $H$ (resp. $T$) denotes all HIV positive (resp. HIV negative) individuals in the population. Let $\pi$ be the probability that a person is HIV positive. Hence, $1 - \pi$ denotes the probability that a person is HIV negative. We want to estimate $\pi$. Note that merely interviewing a randomly selected group of people and calculating the proportion of HIV positives will not yield an accurate estimator of $\pi$ because the subjects may be reluctant to disclose their true type to the interviewer. To eliminate this “evasive answer bias”, Warner (JASA, 1965) proposed the following “randomized response” methodology of interviewing people about sensitive issues: A random sample of $n$ people is drawn from the population, and the interviewer is furnished with a coin for which the probability of obtaining a heads ($H$) is $p$ and the probability of obtaining a tails ($T$) is $1 - p$. $p$ is known to the interviewer and is chosen such that $p \neq 1/2$. In each interview, the interviewee is asked to toss the coin unobserved by the interviewer and report only whether or not the outcome of the toss is a letter to which the interviewee belongs; i.e., the interviewee is only required to say yes or no according to whether or not the coin toss outcome corresponds to the correct group. He does not report the group corresponding to the coin toss outcome. Assume that these yes and no reports are made truthfully. For $i = 1, \ldots, n$ let $X_i$ denote a random variable which takes value one if the $i^{th}$ interviewee says yes, and is zero otherwise. Warner suggests using the random sample $X_1, \ldots, X_n$ to calculate the mle of $\pi$. Answer the following questions:

(i) Find the pmf of $X_i$.

(ii) Use the random sample $X_1, \ldots, X_n$ to calculate $\hat{\pi}$, the mle of $\pi$.

(iii) Show that $\hat{\pi}$ is unbiased and consistent for $\pi$.

This problem set does not have to be turned in.
(iv) Find the asymptotic distribution of $n^{1/2}(\hat{\pi} - \pi)$.
(v) Explain why the restriction $p \neq 1/2$ is necessary for Warner’s suggestion to work.

**Problem 4**

The department of labor wants you to judge the efficacy of a job training program for the unemployed where $n$ persons are randomly selected from the pool of unemployed and go through the program. After the training is over a total of $S_n$ participants receive job offers. Let $\theta$ denote the probability that a person gets a job after having gone through the program. Past experience suggests that a proportion $\theta_0$ of the unemployed will get jobs without going through any training program. To test $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$ you decide to use the following rule: Reject $H_0$ if $S_n \geq k$, where $k$ is some non negative integer.

(i) Write down the critical region for this test?
(ii) Find the exact size of this test.
(iii) What is the exact probability of Type II error?
(iv) Find the power function for this test.

[Hint for (ii), (iii) and (iv): What is the distribution of $S_n$?]

**Problem 5**

Let $X_1, \ldots, X_n$ be a random sample from a Beta($1, \beta$) distribution; i.e.
\[
 f(x) = \beta(1-x)^{\beta-1}, \quad 0 < x < 1, \quad \beta > 0.
\]
Consider testing $H_0 : \beta = 1$ against $H_1 : \beta < 1$.

(i) Show that $-2\sum_{i=1}^n \ln(1-X_i)$ has a Chi-squared distribution under $H_0$. What are the degrees of freedom for this distribution?
(ii) Suppose we decide to reject $H_0$ if $\prod_{i=1}^n (1-X_i) \leq c$. What value of the constant $c$ should be used if $n = 10$ and we want a test with size $\alpha = 0.05$?

**Problem 6**

Let $X_1, \ldots, X_n$ be a random sample from the Bernoulli($p$) distribution and consider testing $H_0 : p = 1/2$ against $p = 1/3$. Suppose we decide to reject $H_0$ if $\sum_{i=1}^n X_i \leq c$. Use the central limit theorem to find $n$ and $c$ so that the probability of Type I error is 0.1 and the probability of Type II error is 0.2.

**Problem 7**

Let $X_1, \ldots, X_n$ be a random sample from an Exponential($\theta$) distribution, while $Y_1, \ldots, Y_n$ is a random sample from an Exponential($\mu$) distribution. The two samples are independent. Find the form of the LRT for testing $H_0 : \theta = \mu$, against $H_1 : \theta \neq \mu$. Do not compute the distribution of the LRT statistic.