STATISTICS 5415 (SPRING 2010)
PROBLEM SET 4

GAUTAM TRIPATHI

PROBLEM 1

Let $X$ be a random variable and $c$ a constant. Find $E(X|c)$ and $E(c|X)$. Comment on the difference, if any, between the two answers.

PROBLEM 2

If $A$ and $B$ are independent events, show that $A^c \perp \perp B$ and $A^c \perp \perp B^c$.

PROBLEM 3

If events $A$ and $B$ satisfy $\text{Pr}(A) \text{Pr}(B) = 0$, show that they are independent.

PROBLEM 4

Show that random variables $X$ and $Y$ are independent if and only if

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

holds for all functions $g$ and $h$.

PROBLEM 5

In class we had shown that $E(Y|X)$ is the best predictor of $Y$ given $X$. In this exercise another optimality property of $E(Y|X)$ is highlighted. Let $a$ and $b \neq 0$ be arbitrary constants and $\mathcal{F}$ denote the set of all functions of $X$. Use the Cauchy-Schwarz inequality to show that $a + bE(Y|X)$ maximizes the correlation between $Y$ and all functions of $X$; i.e.,

$$a + bE(Y|X) \in \arg\max_{f \in \mathcal{F}} |\text{corr}(Y, f(X))|.$$ 

PROBLEM 6

Let $X \sim N(0, 1)$ and $c$ a constant. Suppose we observe data $(Y, D)$, where

$$Y := \begin{cases} 
X & \text{if } X < c \\
n & \text{if } X \geq c 
\end{cases}$$

and $D := 1(X < c)$. Find the joint cdf and density of $(Y, D)$.

Due: February 25, 2010.
Problem 7
Let $N \sim \text{Poisson}(\lambda)$ and $S_N := \sum_{i=1}^{N} X_i$ denote summation with a random index, where $X_1, \ldots, X_N \mid N \sim \text{Bernoulli}(p)$. Find $E S_N$ and $\text{var} S_N$.

Problem 8
Let $X$ be a random variable such that $\Pr(X \in S) > 0$. Show that
\[
E(Y \mid X \in S) = \frac{1}{\Pr(X \in S)} \int_{x \in S} E(Y \mid X = x) \text{pdf}_X(x) \, dx.
\]

Problem 9
Define the conditional covariance between random variables $X$ and $Y$, given $Z$, as $\text{cov}(X, Y \mid Z) := E[(X - E(X \mid Z))(Y - E(Y \mid Z)) \mid Z]$. Show that:
(i) $\text{cov}(X, Y \mid Z) = E(XY \mid Z) - E(X \mid Z)E(Y \mid Z)$.
(ii) $\text{cov}(X, Y) = E\text{cov}(X, Y \mid Z) + \text{cov}(E[X \mid Z], E[Y \mid Z])$; this is analogous to the variance decomposition result we proved in class.