Theorem: Existence of a vNM utility function

Let preferences \( \succsim \) over gambles in \( G \) satisfy axioms G1 to G6 (as presented in class). Then there exists a utility function \( u(g) \) that maps \( G \) on the real line, s.t. \( u(g) \) has the expected utility property.

Proof

Consider a gamble \( g \in G \). Define \( u(g) \) as the probability such that

\[
g \sim (u(g) \circ b_1, (1 - u(g)) \circ b_n),
\]

where \( b_1 \) and \( b_n \) are the best and worst outcome, respectively. Such a \( u(g) \) exists by the continuity axiom G3.

Furthermore, notice that by monotonicity G4, \( u(g) \) is unique. To see this, assume the opposite; i.e., let

\[
g \sim (u(g) \circ b_1, (1 - u(g)) \circ b_n)
\]

and

\[
g \sim (\tilde{u}(g) \circ b_1, (1 - \tilde{u}(g)) \circ b_n)
\]

where \( \tilde{u}(g) > u(g) \). Then by G4

\[
(\tilde{u}(g) \circ b_1, (1 - \tilde{u}(g)) \circ b_n) \succ (u(g) \circ b_1, (1 - u(g)) \circ b_n)
\]

contradicting that we are indifferent between \( g \) and both lotteries.

\( u(g) \) is hence a function, not a multi-valued correspondence.

It remains to show the vNM property and that \( u \) represents \( \succsim \).

To show vNM property:

Let \( g \) be a gamble and let \( g_s = (\alpha_1 \circ b_1, ..., \alpha_n \circ b_n) \) be the simple gamble
induced by $g$. By G6, we know that

$$g \sim (\alpha_1 \circ b_1, ..., \alpha_n \circ b_n).$$

By definition of $u$, we have

$$b_i \sim (u(b_i) \circ b_1, (1 - u(b_i)) \circ b_n).$$

Using the substitution axiom G5, we thus have from the above:

$$g \sim (\alpha_1 \circ (u(b_1) \circ b_1, (1 - u(b_1)) \circ b_n), ..., \alpha_n \circ (u(b_n) \circ b_1, (1 - u(b_n)) \circ b_n)).$$

We can now use G6 (reduction to simple gambles) to write:

$$g \sim \left( \sum_{i=1}^{n} \alpha_i u(b_i) \circ b_1, \sum_{i=1}^{n} \alpha_i [1 - u(b_i)] \circ b_n \right).$$

We therefore see that

$$u(g) = \sum_{i=1}^{n} \alpha_i u(b_i);$$

i.e., $u$ has the vNM property.

To show that $u$ represents $\succsim$:

Let $g_1 \succ g_2$:

$u(g_1)$ s.t. $g_1 \sim (u(g_1) \circ b_1, (1 - u(g_1)) \circ b_n),$

$u(g_2)$ s.t. $g_2 \sim (u(g_2) \circ b_1, (1 - u(g_2)) \circ b_n).$

Thus by monotonicity $u(g_1) > u(g_2)$. Similarly, for $g_2 \succ g_1$, we have $u(g_2) > u(g_1)$ and for $g_1 \sim g_2$ we have $u(g_1) = u(g_2)$. Therefore $u$ represents $\succsim$.

Theorem: Uniqueness of expected utility function:

An expected utility function is unique up to an affine transformation:

Let $u(g)$ represent $\succsim$. Then the vNM utility function $v(g)$ also represents $\succsim$ iff

$$v(g) = a_0 + a_1 u(g)$$

where $a_1 > 0$. 

Proof:

a) $v(g)$ being an affine transformation of $u(g)$ implies $v(g)$ represents $\succeq$

Let $g_1 \succ g_2$. Since $u$ represents $\succeq$, we have $u(g_1) > u(g_2)$ and also for $a_1 > 0$ $a_0 + a_1 u(g_1) > a_0 + a_1 u(g_2)$. Similar reasoning for $g_1 \sim g_2$, establishing that $v(g) = a_0 + a_1 u(g)$ represents $\succeq$.

b) $v(g)$ representing $\succeq$ implies that $v(g)$ is an affine transformation of $u(g)$

Let $g$ be a simple gamble

$$g = (p_1 \circ b_1, \ldots, p_n \circ b_n)$$

where $b_1 \succeq b_2 \succeq \ldots \succeq b_n$.

Since $u$ represents $\succeq$, we have

$$u(b_1) \geq u(b_2) \geq \ldots \geq u(b_n).$$

Assuming $b_1 \succ b_n$, we have $u(b_1) > u(b_n)$. Hence there exists $\alpha_i \in [0,1]$ such that

$$(0.1) \quad u(b_i) = \alpha_i u(b_1) + (1 - \alpha_i) u(b_n).$$

Since $u$ is a vNM utility function, we know that

$$b_i \sim (\alpha_i \circ b_1, (1 - \alpha_i) \circ b_n).$$

Therefore, since $v(g)$ represents $\succeq$, we must have

$$v(b_i) = v(\alpha_i \circ b_1, (1 - \alpha_i) \circ b_n).$$

Since $v(g)$ is a vNM utility function, we must have

$$(0.2) \quad v(b_i) = \alpha_i v(b_1) + (1 - \alpha_i) v(b_n).$$

From (0.1), we get

$$\frac{u(b_1) - u(b_i)}{u(b_i) - u(b_n)} = \frac{u(b_1) - \alpha_i u(b_1) - [1 - \alpha_i] u(b_n)}{\alpha_i u(b_1) + [1 - \alpha_i] u(b_n) - u(b_n)}$$

$$\frac{u(b_1) - u(b_i)}{u(b_i) - u(b_n)} = \frac{(1 - \alpha_i)[u(b_1) - u(b_n)]}{\alpha_i[u(b_1) - u(b_n)]}$$
\[ \frac{u(b_1) - u(b_i)}{u(b_i) - u(b_n)} = \frac{1 - \alpha_i}{\alpha_i}. \]

From (0.2), we get
\[ \frac{v(b_1) - v(b_i)}{v(b_i) - v(b_n)} = \frac{v(b_1) - \alpha_i v(b_1) - [1 - \alpha_i] v(b_n)}{\alpha_i v(b_1) + [1 - \alpha_i] v(b_n) - v(b_n)}. \]
\[ v(b_1) - v(b_i) = \frac{(1 - \alpha_i) [v(b_1) - v(b_n)]}{\alpha_i [v(b_1) - v(b_n)]}. \]
\[ \frac{v(b_1) - v(b_i)}{v(b_i) - v(b_n)} = \frac{1 - \alpha_i}{\alpha_i}. \]

Hence
\[ \frac{u(b_1) - u(b_i)}{u(b_i) - u(b_n)} = \frac{v(b_1) - v(b_i)}{v(b_i) - v(b_n)}, \]
\[ [u(b_1) - u(b_n)] v(b_i) = [u(b_i) - u(b_n)] v(b_1) + [u(b_1) - u(b_i)] v(b_n), \]
\[ v(b_i) = \frac{u(b_1) v(b_n) - v(b_1) u(b_n)}{u(b_1) - u(b_n)} + \frac{v(b_1) - v(b_n)}{u(b_1) - u(b_n)} u(b_i). \]

Notice that this shows that
\[ v(b_i) = a_0 + a_1 u(b_i) \]
where \( a_1 > 0 \). Hence \( v(b_i) \) is an affine transformation of \( u(b_i) \).

Therefore by the vNM property of \( v \):
\[ v(g) = \sum_{i=1}^{n} p_i v(b_i) \]
\[ v(g) = \sum_{i=1}^{n} p_i [a_0 + a_1 u(b_i)] \]
\[ v(g) = a_0 \sum_{i=1}^{n} p_i + a_1 \sum_{i=1}^{n} p_i u(b_i) \]
\[ v(g) = a_0 + a_1 u(g) \]