ECONOMICS 6201 (FALL 2009)
PROBLEM SET 4 - SOLUTIONS

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PROBLEM A

(a) proof that \( U(x) \) quasiconcave implies convexity of \( \succsim \)
Since \( U(x) \) is quasiconcave, we have \( U(tx_1 + (1-t)x_2) \geq \min(U(x_1), U(x_2)) \). Without loss of

generality, let \( x_1 \succsim x_2 \), so \( U(x_1) \geq U(x_2) \). Then by quasiconcavity, we have \( U(tx_1 + (1-t)x_2) \geq U(x_2) \), hence \( tx_1 + (1-t)x_2 \succsim x_2 \) what had to be shown.

(b) proof that \( \succsim \) convex implies quasiconcavity of \( \U(x) \)
Without loss of generality, let \( x_1 \succsim x_2 \). Since \( \succsim \) is convex, \( tx_1 + (1-t)x_2 \succsim x_2 \). Since \( \U \) represents \( \succsim \), we have \( U(x_1) \geq U(x_2) \) and \( U(tx_1 + (1-t)x_2) \geq U(x_2) \). Hence \( U(tx_1 + (1-t)x_2) \geq \min(U(x_1), U(x_2)) \), i.e. \( \U(x) \) is quasiconcave.

PROBLEM B

The Lagrange function for Emma’s problem is
\[
L = x_1 + \ln(x_2) + \lambda(T - x_1 - x_2).
\]

The first-order conditions are
\[
\begin{align*}
1 - \lambda &= 0, \\
\frac{1}{x_2} - \lambda &= 0, \\
T - x_1 - x_2 &= 0.
\end{align*}
\]

From the FOC, it is easy to find the interior solution as \( x_2 = 1 \) and \( x_1 = T - 1 = 4 \). However,

this solution is only valid if \( T \geq 1 \). If \( T < 1 \), Emma only goes trick-or-treating (\( x_2 = T \)) and does not watch any horror movies (\( x_1 = 0 \)). Summarizing, for \( T < 1 \), \( x_1 \) does not depend on the time amount whereas \( x_2 \) increases with \( T \). In contrast, for \( T \geq 1 \), \( x_2 \) does not depend on \( T \), whereas \( x_1 \) is increasing in \( T \).

PROBLEM C

In a first step, show that the expenditure function is linear in \( u \): \( \E(p,u) = \min_x[p \cdot U(x) \geq u] \)
\[
\begin{align*}
\E(p,u) &= \min_x[p \cdot U(x) \geq u] \\
\E(p,u) &= \min_x[p \cdot U(x) \geq 1] \\
\E(p,u) &= u \cdot \min_x[p \cdot U(x) \geq 1] \\
\E(p,u) &= e(p)u.
\end{align*}
\]

In a second step, use the unnamed lemma (the cousin of Shephard’s lemma) to get:
\[
h_i(p,u) = \frac{\partial E}{\partial p_i} = \frac{\partial e}{\partial p_i}u.
\]

Defining \( a_i(p) = \frac{\partial e}{\partial p_i} \) and stacking up the demands for the goods to obtain the demand vector, we thus have that \( h(p,u) = a(p)u \).
PROBLEM D

The quantity index is $\left[\sum_{i=1}^{n} x_i^{\alpha}\right]^\frac{1}{\alpha}$. To find the price index, set up the Lagrange function for the expenditure minimization problem when utility is 1. The Lagrange function is:

$$\sum_{i=1}^{n} p_i x_i + \lambda \left( 1 - \sum_{i=1}^{n} x_i^{\alpha} \right)^{\frac{1}{\alpha}}$$

The first-order conditions are

$$p_j - \lambda x_j^{\alpha-1} \left( \sum_{i=1}^{n} x_i^{\alpha} \right)^{\frac{1-a}{\alpha}} = 0$$

for all $j$ and

$$1 = \left( \sum_{i=1}^{n} x_i^{\alpha} \right)^{\frac{1}{\alpha}}.$$ 

The first FOC can be solved for $x_j^{\alpha}$. Using also the information from the second FOC, we have

$$x_j^{\alpha} = p_j^{\frac{1}{\alpha}} \lambda^{\frac{-1}{\alpha}}.$$ 

We can sum this expression over all $j$ and solve for $\lambda$ to find

$$\lambda = \left( \sum_{i=1}^{n} p_i^{\frac{1}{\alpha}} \right)^{\frac{-1}{\alpha}}.$$ 

Substituting back into $x_j$, we have

$$x_j = p_j^{\frac{1}{\alpha}} \left( \sum_{i=1}^{n} p_i^{\frac{1}{\alpha}} \right)^{-\frac{1}{\alpha}}.$$ 

Plugging this back into $\sum_{i=1}^{n} p_i x_i$, we have found the price index as

$$P = \left( \sum_{i=1}^{n} p_i^{\frac{1}{\alpha}} \right)^{\frac{a}{\alpha}}.$$ 

PROBLEM E

(a) (3 points) The statement is true. A homothetic utility function is defined as a strictly increasing transformation of a linearly homogeneous utility function. The square root function is a strictly increasing function. The function $x_A + 2x_B$ is linearly homogeneous since $\alpha x_A + 2(\alpha x_B) = \alpha(x_A + 2x_B)$. Hence, the given utility function is homothetic.

(b) (10 points) The utility maximization problem is

$$\max_{x_A, x_B} \left\{ \sqrt{x_A + 2x_B} | p_A x_A + p_B x_B \leq m \right\}.$$ 

The utility function is actually linear in its structure (this is easy to see since the indifference curve for a utility level $u^2$ is linear). In this case, the consumer should buy the good of which an efficiency unit is cheapest. The corresponding indirect utility function is hence

$$V(p, m) = \sqrt{\frac{m}{\min\{p_A, p_B\}}}.$$ 

Solving for $m$ gives the expenditure function as

$$E(p, u) = u^2 \min\{p_A, \frac{p_B}{2}\}.$$ 

(c) (3 points) Using the previous results, the money metric indirect utility function is

$$E(p, V(q, m)) = \frac{m}{\min\{q_A, \frac{q_B}{2}\}} \min\{p_A, \frac{p_B}{2}\}.$$
(d) (4 points) Batman’s utility level in New Haven is

\[ u = \sqrt{\frac{2000}{\min\{20, \frac{60}{2}\}}} = 10. \]

To obtain the same level of utility in Storrs, he needs an income of

\[ \mu = (10)^2 \min\{10, \frac{20}{2}\} = 1000. \]