GENERAL EQUILIBRIUM THEOREMS IN THE
TWO-AGENT-TWO-GOOD EXCHANGE ECONOMY

In this section, we are going to look at some theorems for GE models and prove them for our Edgeworth box economy. It is possible (look at the books) to also do the proofs for much more complicated economies with many goods, many individuals and production.

Definition: Excess demand function
The excess demand function for a good in our two-person economy is defined as

\[ z_x = x_1 - \bar{x}_1 + x_2 - \bar{x}_2, \]
\[ z_y = y_1 - \bar{y}_1 + y_2 - \bar{y}_2, \]

for goods \( x \) and \( y \), respectively. The excess demand functions depend on prices and endowments.

You should notice that \( x_i \) and \( y_i \) are Marshallian demands and thus depend on the price vector and the value of endowments (inner product of price and endowment vector) that an individual possesses. Since the Marshallian demands are homogeneous of degree 0 in price and income, this means for our endowment economy that the Marshallian demands are homogeneous of degree 0 in prices. Also, Marshallian demands are continuous for strictly convex preferences. Similarly, the excess demand vector \((z_x, z_y)\) is homogeneous of degree 0 in prices and continuous.

In the following, we are usually going to assume that if the price of a good is zero, then we will have positive excess demand for this good (assumption of desirability). Goods for which we have an excess supply at zero price are called free goods.
Walras’ Law
For any price vector \((p_x, p_y)\), we have the inner product of price and excess demand vector equal to zero, so in our economy:

\[ p_x z_x + p_y z_y = 0. \]

Proof:
Plug in the excess demands into the equation that we want to prove:

\[ p_x (x_1 - \bar{x}_1 + x_2 - \bar{x}_2) + p_y (y_1 - \bar{y}_1 + y_2 - \bar{y}_2) = \]

\[ p_x x_1 + p_y y_1 - (p_x \bar{x}_1 + p_y \bar{y}_1) + p_x x_2 + p_y y_2 - (p_x \bar{x}_2 + p_y \bar{y}_2) = 0 \]

since the budget constraints of the two individuals bind.

Walras’ law implies for our two-good framework that if the excess demand for one market is zero, then either the excess demand of the other good is zero as well, or, if the excess demand of the other good is not zero, then its price must be zero.

Definition: unit price simplex
In our model, the unit price simplex \(S\) consists of all price vectors \((p_x, p_y)\) such that \(p_x + p_y = 1\). Notice that this is just another form of price normalization (previously, we chose one price to be equal to one) and completely innocuous given that the Marshallian demands in our framework are homogeneous of degree 0 in prices.

Brouwer fixed-point theorem Let \(f : X \to X\), where \(X \in \mathbb{R}^n\) is a nonempty, compact, and convex set, be a continuous function. Then there exists some \(x \in X\) for which \(f(x) = x\).

We are going to use the Brouwer fixed-point theorem in the following proof of the existence of Walrasian equilibria.

1. **Theorem: Existence of Walrasian equilibria**

Let \((z_x, z_y)\) be a continuous function that maps \((p_x, p_y) \in S\) into \(R^2\) and satisfies Walras law. Then there exists some \((p_x^*, p_y^*) \in S\) such
that \( z_x(p_x^*, p_y^*) \leq 0 \) and \( z_y(p_x^*, p_y^*) \leq 0 \), i.e. there is no excess demand of any good.

**Proof:**

Define the following mapping \( g = (g_x(p_x, p_y), g_y(p_x, p_y)) \):

\[
g_x = \frac{p_x + \max(0, z_x(p_x, p_y))}{1 + \max(0, z_x(p_x, p_y)) + \max(0, z_y(p_x, p_y))},
\]

\[
g_y = \frac{p_y + \max(0, z_y(p_x, p_y))}{1 + \max(0, z_x(p_x, p_y)) + \max(0, z_y(p_x, p_y))}.
\]

You can think of the \( g \) function as some kind of auctioneer who adjusts prices as long as there are still excess demands. As you can see, the function \( g \) maps from the unit simplex into the unit simplex since \( g_x + g_y = 1 \). Therefore, by Brouwer’s fixed-point theorem, there exists \( (p_x^*, p_y^*) \) such that \( (p_x^*, p_y^*) = (g_x, g_y) \). We are now going to show that the fixed point is a Walrasian equilibrium, i.e., there are no excess demands. Using \( (p_x^*, p_y^*) = (g_x, g_y) \), we rewrite the above equations as:

\[
p_x^*[1 + \max(0, z_x(p_x^*, p_y^*)) + \max(0, z_y(p_x^*, p_y^*))] = p_x^* + \max(0, z_x(p_x^*, p_y^*)),
\]

\[
p_y^*[1 + \max(0, z_x(p_x^*, p_y^*)) + \max(0, z_y(p_x^*, p_y^*))] = p_y^* + \max(0, z_y(p_x^*, p_y^*)).
\]

Simplifying yields:

\[
p_x^*\max(0, z_x(p_x^*, p_y^*)) + \max(0, z_y(p_x^*, p_y^*)) = \max(0, z_x(p_x^*, p_y^*)),
\]

\[
p_y^*\max(0, z_x(p_x^*, p_y^*)) + \max(0, z_y(p_x^*, p_y^*)) = \max(0, z_y(p_x^*, p_y^*)).
\]

Multiplying by \( z_x \) and \( z_y \), respectively, and adding up:

\[
[z_x(p_x^*, p_y^*)p_x^* + z_y(p_x^*, p_y^*)p_y^*][\max(0, z_x(p_x^*, p_y^*)) + \max(0, z_y(p_x^*, p_y^*))] =
\]

\[
z_x(p_x^*, p_y^*)\max(0, z_x(p_x^*, p_y^*)) + z_y(p_x^*, p_y^*)\max(0, z_y(p_x^*, p_y^*)).
\]

From Walras law, we know that \( z_x(p_x^*, p_y^*)p_x^* + z_y(p_x^*, p_y^*)p_y^* = 0 \), therefore \( z_x(p_x^*, p_y^*)\max(0, z_x(p_x^*, p_y^*)) + z_y(p_x^*, p_y^*)\max(0, z_y(p_x^*, p_y^*)) = 0 \). But for this result to hold, we must have \( z_x(p_x^*, p_y^*) \leq 0 \) and \( z_y(p_x^*, p_y^*) \leq 0 \), what we had to show.
We are now going to restate the definition of Walrasian equilibrium:

2. Definition of Walrasian Equilibrium

Assume that all goods are desirable, i.e., there are no free goods. An allocation pair \((p^*_x, p^*_y, x^*_1, y^*_1, x^*_2, y^*_2)\) is a Walrasian equilibrium if

1. the allocation is feasible: \(x^*_1 + x^*_2 = \bar{x}_1 + \bar{x}_2\) and \(y^*_1 + y^*_2 = \bar{y}_1 + \bar{y}_2\),
2. the agent is making an optimal choice from his budget set: If \((x'_i, y'_i)\) is preferred by individual \(i\) over \((x^*_i, y^*_i)\), then it must be that \(p^*_x x'_i + p^*_y y'_i > p^*_x \bar{x}_i + p^*_y \bar{y}_i\), i.e. the preferred allocation is not individually feasible for the agent.

We already talked about the fact that the Walrasian equilibrium is Pareto efficient (first theorem of welfare economics) and about the fact that any Pareto efficient allocation can be supported as a Walrasian equilibrium if wealth is redistributed accordingly (second theorem of welfare economics).

A simple version of the proofs runs as follows:

3. First Theorem of Welfare Economics

If \((p^*_x, p^*_y, x^*_1, y^*_1, x^*_2, y^*_2)\) is a Walrasian equilibrium, then \((x^*_1, y^*_1, x^*_2, y^*_2)\) is a Pareto efficient allocation.

**proof:**
Suppose not, i.e. there exists a feasible allocation \((x'_1, y'_1, x'_2, y'_2)\) such that \((x'_1, y'_1) \succ (x^*_1, y^*_1)\) and \((x'_2, y'_2) \succ (x^*_2, y^*_2)\). Since \((p^*_x, p^*_y, x^*_1, y^*_1, x^*_2, y^*_2)\) is a Walrasian equilibrium, we know that \((x'_i, y'_i)\) was not individually feasible for individual \(i\), i.e. \(p^*_x x'_i + p^*_y y'_i > p^*_x \bar{x}_i + p^*_y \bar{y}_i\) where \(i = 1, 2\). Summing these conditions up, we find

\[ p^*_x (x'_1 + x'_2) + p^*_y (y'_1 + y'_2) > p^*_x (\bar{x}_1 + \bar{x}_2) + p^*_y (\bar{y}_1 + \bar{y}_2). \]

But by feasibility of \((x'_1, y'_1, x'_2, y'_2)\) we also know that \(x'_1 + x'_2 = \bar{x}_1 + \bar{x}_2\) and \(y'_1 + y'_2 = \bar{y}_1 + \bar{y}_2\), a direct contradiction to the previous statement.
4. SECOND THEOREM OF WELFARE ECONOMICS

Suppose that \((x_1^*, y_1^*, x_2^*, y_2^*)\) is a Pareto efficient allocation and that preferences are nonsatiated. Suppose further that a competitive equilibrium from the initial endowments \((x_1^*, y_1^*, x_2^*, y_2^*)\) exists (this requires continuity of aggregate demand) and let it be given by \((p'_y, p'_x, x'_1, y'_1, x'_2, y'_2)\). Then \((p'_y, p'_x, x_1^*, y_1^*, x_2^*, y_2^*)\) is a competitive equilibrium.

proof:
Since \((x_1^*, y_1^*)\) and \((x_2^*, y_2^*)\) are the endowments, they are certainly affordable for individuals 1 and 2, respectively. Therefore, by existence of a competitive equilibrium, we know that \((x'_1, y'_1) \succ (x_1^*, y_1^*)\) and \((x'_2, y'_2) \succ (x_2^*, y_2^*)\). But since \((x_1^*, y_1^*, x_2^*, y_2^*)\) is Pareto efficient, we also know that \((x'_1, y'_1) \sim (x_1^*, y_1^*)\) and \((x'_2, y'_2) \sim (x_2^*, y_2^*)\), hence \((p'_y, p'_x, x_1^*, y_1^*, x_2^*, y_2^*)\) is a Walrasian equilibrium.