The Effects of Credit Risk on Dynamic Portfolio Management: A New Computational Approach

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Working Paper 2009-03

January 2009
Abstract
The study investigates the role of credit risk in a continuous time stochastic asset allocation model, since the traditional dynamic framework does not provide credit risk flexibility. The general model of the study extends the traditional dynamic efficiency framework by explicitly deriving the optimal value function for the infinite horizon stochastic control problem via a weighted volatility measure of market and credit risk. The model’s optimal strategy was then compared to that obtained from a benchmark Markowitz-type dynamic optimization framework to determine which specification adequately reflects the optimal terminal investment returns and strategy under credit and market risks. The paper shows that an investor's optimal terminal return is lower than typically indicated under the traditional mean-variance framework during periods of elevated credit risk. Hence I conclude that, while the traditional dynamic mean-variance approach may indicate the ideal, in the presence of credit-risk it does not accurately reflect the observed optimal returns, terminal wealth and portfolio selection strategies.

Journal of Economic Literature Classification: G0, G10, C02, C15

Keywords: Dynamic Strategies; Credit Risk; Mean-Variance Analysis; Optimal Portfolio Selection; Viscosity Solution; Credit Default Swaps; Default Risk; Dynamic Control
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1.0 Introduction

Managers of financial assets, such as stocks and bonds, typically seek to maximize their expected returns on investment for a given level of risk. In fact the ultimate goal of modern investment theory is to construct an optimal portfolio of investments from a set of risky assets. Investment in these risky assets generally depends on the discount rate, the market return and the volatility parameters. From the classical pedagogical work of Markowitz (1959), the optimal portfolio for a given level of risk and set of constraints can be derived under the "mean-variance (MV) efficiency frontier" using known optimization algorithms such as quadratic programming.

Under any given mean-variance optimization framework, the portfolio with the highest expected risk adjusted return and the smallest minimized variance (market risk) is said to be more efficient. These properties of efficient portfolios are central to both static and dynamic optimization. However, a recognizable weakness in this methodology is the apparent exclusion of a credit risk measure from the investor’s operational risk frontier. As investors continually seek higher returns on investments it becomes increasingly difficult to ignore the effects of credit risk on expected portfolio returns. In fact, in any dynamic

\[ u^* - p_{cds} = u^*_h \text{ where } u^*_h < u^* \]

Here credit risk is defined as the risk of default or the deterioration in credit quality of a reference entity that is part of the portfolio that the investor holds. For further reading on credit risk the reader may look at Jarrow et al (2001), Duffie (1999), and Dunbar (2008).

While complete hedging is not possible, the investor could buy credit default swaps (CDS) as a hedge against credit quality changes (risk) in the debt component of the investment portfolio opportunity set, however the cost of the CDS premium would ultimately result in a net reduction in the overall terminal hedged payoffs, because the terminal hedged payoff would be lower by the cost of the hedge.

\[ u^* - p_{cds} = u^*_h \text{ where } u^*_h < u^* \]
portfolio management setting, since an infinitely lived agent may hold his portfolio \( \sum_{i=1}^{n} W_i \) over some investment horizon \( \tau > t \), the implied credit risk of holding the securities in the portfolio increases with the investment horizon. As such, an ideal asset in period \( t_0 \) may disappear (default) prior to the intended maturity period \( \tau > t \), which poses some level of risk to the agent’s expected level of return.

Therefore as discussed later in section 2.0, this paper examines the implication of credit risk in dynamic portfolio optimization by developing a baseline dynamic optimization model and an extended credit-risk dynamic optimization model that were both used to investigate optimal investment selection strategies and terminal returns under credit and market risk. More specifically the study derived; (a) closed-form solutions for optimal asset allocation and investment strategies in both a mean-market risk and a mean-market/credit risk framework via an optimal value function that satisfies some infinite horizon stochastic control function; and (b) the reaction function of the risk-averse agent when faced with both market and credit risk. Not surprising the solution of the Hamilton-Jacobi-Bellman (HJB) equations illustrate that given some added investment (credit) risk an agent will react by modifying his efficient strategy of portfolio selection resulting in a lower optimal terminal return relative to the benchmark strategy.

The baseline model which is the prevailing approach in dynamic portfolio optimization is used to demonstrate the agent’s risk aversion, expected returns and portfolio selection strategy given market risk. Later the more flexible extended framework is then used to show the agent’s modified response given a more fulsome risk measure that comprises credit risk. This proposed enhanced credit-risk framework assumes that investment risk is represented by a risk continuum that may initially subject the portfolio’s
assets to market risk, which, depending on economic and credit market conditions could lead to an increase in the asset’s risk of default thereby exposing the investor to credit risk.

The results of the study’s empirical analyses illustrate that the exclusion of credit risk from any portfolio optimization analysis may overstate the familiar risk frontier by overstating the investor’s optimal terminal investment returns ($u^*$) during periods of elevated credit risks, because existing models implicitly assume the non-existence of credit events.\(^4\) In fact the study’s empirical analyses demonstrate that the investor’s true terminal return may lie on a given risk frontier or in the region that is bordered by both the benchmark and the credit-risk enhanced models illustrated in figure 1. The upper boundary represents the market-risk frontier which nests the extended model when credit risk converges to zero, while the lower boundary is the optimal frontier that exists when the investor is faced with both market and credit risk.

In deriving the study’s extended model we start from the basic formulation of the Markowitz (1959) framework via a linear quadratic control model where we denote the returns of \(n\) risky assets by a \(n*1\) vector \(\mathbf{R}\), the unobserved future return \(r\) (assumed unknown), and the instantaneous variance-covariance matrix of stock and bond fund returns;

\[
\sum w_i E(r_i) = \mu \quad \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \text{cov}(i, j) = \Gamma = \begin{pmatrix}
\sigma_i^2 & \sigma_i \sigma_j \\
\sigma_i \sigma_j & \sigma_j^2
\end{pmatrix}
\]  

\[\text{(1.1)}\]

\(^4\) A credit event is defined as a sudden progressive change in an asset’s credit standing, brought on by events such as a default or bankruptcy that raises doubts about the asset’s ability to repay its future obligations or payoff.
where \( \text{cov}(i, j) \) is the covariance between returns from investments in \( i \) and \( j \). The mean-variance optimization solves the asset allocation \( w_i \), which minimizes the portfolio risk \( \sigma_p^2 \), while achieving a certain target return \( \mu^* \). Thus our problem is to minimize

\[
\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \text{cov}(i, j)
\]

Subject to the constraints

\[
\sum_{i=1}^{n} w_i \mu_i \geq \mu^* \quad \text{and} \quad w_1 + w_2 + \ldots + w_p = 1
\]

where the portfolio strategies are defined by their weights \( w_1 + \ldots + w_p \), the short sales constraint \( \psi_i \leq w_i \leq \eta_i \ (i = 1, 2, \ldots, n) \), and where we use \( \sigma_p^2 \) to represent a pooled cross-sectional measure of credit and market risk in the extended model framework. In the traditional framework \( \sigma_p^2 \) represents market risk.

Expression 1.2 demonstrates that under the extended framework, when an investor makes an investment in a firm, the investor is exposed to both market and credit risk. Finance theory and recent empirical work by Dunbar (2008) suggests that market and credit risk are intrinsically related to each other. In fact on the “investment risk continuum”, when the value of the firm’s assets unexpectedly changes, market risk is created, which increases the probability of default, subsequently generating credit risk. This underscores the need for the optimization framework to investigate the dual impacts of both credit and market risk on expected returns.
In addition, the paper illustrates that the true short-run dynamic optimum may be on a lower portfolio efficiency curve\(^5\) than previously thought because of the agent’s level of credit-risk aversion. Expression 1.4, demonstrates that as the agent’s risk aversion increases the expected optimal return declines. In fact as financial markets experience a “flight to quality,” which typically characterizes periods of high credit risk, the rational agent will receive a lower total return because of this lower portfolio efficiency curve, as depicted in figure 1.

**Proposition 1.** *The yield of the optimal payoff is on the mean-variance frontier with greater investment in risky assets for investors with lower risk aversion.*

\[
\begin{align*}
  u^* &= u^f e^{\int_r^{\hat{u}} ds} (\sigma_p^2, \phi(\bullet)) + \frac{1}{\varphi} \left( \hat{u} - u^f e^{\int_r^{\hat{u}} ds} (\sigma_p^2, \phi(\bullet)) \right) \\
  &= \frac{d - W_0 e^{\int_r^{\hat{u}} ds}}{1 - e^{\int_r^{\hat{u}} ds}} \\
  &= \frac{d - W_0 e^{\int_r^{\hat{u}} ds}}{1 - e^{\int_r^{\hat{u}} ds}} (1.4)
\end{align*}
\]

where \(\varphi\) is the investor’s coefficient of risk aversion, \(\phi(\bullet)\) is the cumulative distribution function,\(^6\) \(u^f\) is the risk-free rate and \(\hat{u}\) is the average market payoff. The right hand side of the closed form expression in 1.4 is derived and discussed in section 4. From proposition 1, the investor with the lower risk aversion will have a target terminal return \(\left(u^*\right)\) that is on a higher, or on the upper part of the risk frontier depicted in Figure 1, such as a point \(c\), rather than the minimum second moment yield which is on the lower part of the frontier. Note that when \(\varphi\) is small, the investor cares more about small deviations from the target terminal return, while for large \(\varphi\) the investor is more concerned with larger deviations from the target return, as he penalizes these deviations more heavily.

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\(^5\)This is because of the assumption that investors face both credit and market risks in their investment choices.

\(^6\)Since asset prices are generally not normally distributed the study proposes a polynomial expansion of the log-normal density function known as the Grams-Charlier expansion because of the ability to incorporate the data’s skewness and kurtosis in the function.
The remainder of the paper is organized into four sections. In section 2 we introduce the market-credit risk framework and discuss approaches in portfolio optimization. Section 3 lays out the basic setup of the model investigated in this paper. This section introduces the dynamic framework of the model and discusses the technical background for optimal dynamic asset allocation, giving some overview of current dynamic asset allocation methodologies and the analytical procedure for including the credit risk proxy to the optimization process. Section 4 derives the optimal portfolio problem under credit risk. The model is later calibrated to U.S. postwar interest rate, stock return, credit and market risk data. Section 5 presents the data, some representative calculations and discussions on the main empirical findings regarding the role of credit
risk in the dynamic mean-variance framework. Section 6 summarizes the finding and proposes areas of future research.

2.0 Approaches to Dynamic Asset Allocation

The literature on portfolio management has focused in recent years on asset allocation using either a static or dynamic framework based on Markowitz’s celebrated mean-variance analyses. Markowitz’s earlier work was followed by a number of researchers who were primarily concerned with analyzing the investor’s optimum allocation between equity assets and cash balances when returns are independent and identically distributed\(^7\) (Latane and Tuttle (1967), Merton (1969), Samuelson (1969) and Hakansson (1970)). Since Markowitz’s 1959 publication, one of the most spectacular breakthroughs in stochastic portfolio theory was from Merton (1971, 1973), who derived optimal dynamic portfolio allocation in continuous time where security prices were allowed to follow a diffusion process. However, despite this and other fundamental work by Merton, relatively little was done until revived by recent empirical work demonstrating asset return predictability in the late 1990s.

Brennan et al (1997) analyzed the portfolio problem of long-lived investors in stock, bonds and money balances under stochastic variations in interest rates and a predictable equity premium. Kim and Omberg (1996) analyzed the optimal strategy of an investor when interest rates are constant but the equity premium follows an Ornstein-Uhlenbeck process. Later, Omberg (1999), Brennan and Xia (2000) and Sorenson (2000) derive optimal dynamic strategies when the interest rate follows a Vasicek (1977) process

\(^7\) More recently researchers have been relaxing the independent and identically distributed (i.i.d.) assumption because return days are not i.i.d. in practice because of the discrete breaks of weekends.
and risk premia are constant. In a study closely related to this paper, Bajeux and Portait (1998) conclude that the dynamic efficient frontier derived from dynamically managed optimal portfolio strategies outperform the standard Markowitz (1959) frontier derived from efficient static portfolio strategies. Moreover, while this paper is also closely related to a number of others in the literature, none of these papers allows for a cross-sectional pooled (market-credit) risk measure that simulates the agent’s true investment risk frontier.

Over the years various researchers have highlighted the drawbacks of the volatility risk measure in the classical mean-variance approach. So as expected, a number of alternative risk measures have been proposed over the years. Markowitz proposed a semi-variance measure which was later used in Markowitz et al (1993) to derive an optimal asset allocation. Another popular alternative to emerged during this time period was the down-side risk framework which suggests that investors cared more about the risks associated with downside movements\(^8\) than topside risks. However in the real world this approach may appear impractical as it is difficult to realistically isolate downside risk from topside risks as one sometimes lead to the other.

The dynamic asset allocation problem studied in this paper is built upon the Merton (1971) portfolio choice model and the Bajeux et al (1998) dynamic efficient frontier model, with a lower strictly positive bound on real asset prices and wealth, and with an additional \(\hat{I}_t\) process for the price level in the economy. The purpose of this study as laid out in the succeeding paragraph is to demonstrate that the investor’s cross-sectional risk adjusted gains may be lower than indicated by the standard dynamic mean-variance framework, and the traditional mean-variance framework returns may represent

\(^8\) The reader may see for instance Bawa and Lindenberg (1977).
the ideal and not necessarily the observed terminal returns. Observed investor attitudes suggest that investors show aversion to both market and credit risk. For example during the height of one of the most severe credit crisis of this decade, Bear Stearns and MBIA equity may have initially exhibited relatively low levels of market risk (volatility). However investors’ perception of the firm’s credit risk as reflected in the meteoric rise in the price of the firms credit default swap\(^9\) (CDS) resulted in a number of investors abandoning the firms’ equity and debt products prior to their near collapse.\(^{10}\) See figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Quarterly Price Changes for MBIA Credit Default Swap for the period 2003 - 2008}
\end{figure}

Credit-default swaps (CDS) are financial instruments underwritten on bonds and loans that are used to speculate on a company’s ability to repay its debt (credit risk). They pay the buyer face value in exchange for the underlying securities or the cash equivalent should a borrower fail to adhere to its debt agreements. A rise (decline) in the price of the CDS indicates deterioration (increase) in the perception of credit risk. Increased volatility in the price of the credit default swap provides an indication of the level of credit risk in a particular referenced entity.

\(^9\) The 2007 credit crisis created by the elevated volatility in the U.S. credit market spread uncertainty and apprehension among market participants in many countries including some emerging markets. This resulted in an unprecedented “flight to quality” by a number of investors, where they reallocated their investment portfolios away from perceived credit risky assets into more secured lower yielding products such as U.S. Treasury and Municipal Bonds.
The study contributes to the extant literature and its main innovations involve (a) the development of a probability weighted pooled credit-market risk measure for use in deriving the agent’s optimal dynamic asset allocation, (b) an evaluation of the investor’s optimal portfolio strategy under market and credit risk conditions that are based on a quadratic utility maximization, (c) derivation of the efficient portfolio selection strategy in the presence of credit/market risk and (d) deriving a convenient approach for determining the agent’s risk aversion coefficient.

As discussed in section 4 it is assumed that the cross-sectional pooled risk $$\left( \text{var}(\sigma^2_d) dZ_t \right)$$ is a probability weighted average of market risk $$\left( \sigma^2_m \right)$$ and credit risk $$\left( \sigma^2_c \right)$$. In addition, we assume that the cross-sectional pooled risk model nests the familiar mean-variance dynamic asset allocation model. Under conditions of diminishing credit risk the extended cross-sectional pooled risk (CSPR) model converges to the study’s benchmark Markowitz dynamic mean-variance framework.

To illustrate the analytical flexibility and potential of the extended dynamic methodology, an empirical specification was tested under a number of scenarios involving credit and market risk experiences to see how closely the results reflect actual market conditions. The study adopts changes in historical CDS bid-ask spreads as a proxy of credit risk. This is in keeping with the approach by a number of studies in the literature that have used CDS spreads as determinants of default risk, such as Longstaff et al (2005) and Das and Hanouna (2006), Dunbar (2008).
3.0 The Model Structure: Dynamic Framework and Technical Background

This section contains the basic setup of the extended dynamic optimization model investigated in this paper. As discussed in section 1.0, the study develops two alternative models that were used as the primary tools for investigating the investor’s optimal dynamic solution given both market and credit risk. We first develop a benchmark model that allows us to determine the investor’s attitude to market risk; next we creatively exposed the agent to credit risk through a more fulsome risk measure so as to determine any changes in investor’s attitudes to credit risk. We follow the usual conditions for a dynamic portfolio optimization strategy\(^\text{11}\) where the risky security is allowed to follow a geometric Brownian motion and a constant risk-free rate.

**DEFINITIONS.** (i) Credit risk is measured by changes in the credit default swap (CDS) of each firm. (ii) In a short sale, an investor sells borrowed shares in the hope of profiting by buying them back later at a lower price.

The study considers a pure exchange, frictionless economy with a finite horizon \([0, \tau]\) for a fixed \([\tau > 0]\). Following the usual conditions of portfolio optimization, trading can be discrete or continuous and traded are equity products as are both defaultable and default-free zero coupon bonds of all maturities. The portfolio of U.S. Treasury bonds serves as the numeraire. The underlying uncertainty in the economy is represented by a fixed filtered complete probability space \((\Omega, F, \mathbb{P}, \{F^B_t\}_{t \geq 0})\) on which is defined a standard \(\{F^B_t\}_{t \geq 0}\) adapted \(B\)-dimensional Brownian motion \(W(t) \equiv (W^1(t), \ldots, W^m(t))^\top\) (Duffie 1992). The probability space \((\Omega, F, \mathbb{P})\) with the

\(^{11}\) In a dynamic context we construct mean-variance efficient portfolios by optimally allocating wealth across securities as the expected returns and variance-covariance changes over time. As discussed in footnote 2 we may hedge the change in the investment opportunity set, however the hedged payoffs will be lower because of credit risk
filtration $\{F^a_t \mid t \leq a \leq \tau\} (-\infty \leq t < \tau \leq +\infty)$, Hilbert space $H$ equipped with the inner product $\langle \cdot , \cdot \rangle$ and a Euclidean norm $\| \cdot \|_H$, defines the Banach space. Now given the general constrained linear specification;

$$L^2_F (0, \tau, H) = \{ \varphi(*) \text{is an } F_t \text{-- adapted, } H \text{-- valued measurable process on } [\tau, t] \text{ and } E \int_\tau^t \| \varphi(\tau, w) \|^2_H dt < +\infty \}$$

(3.1)

With Euclidean norm;

$$\| \varphi(*) \|_{F, 2} = \left( E \int_\tau^t \| \varphi(\tau, w) \|^2_H dt \right)^{\frac{1}{2}} < +\infty$$

(3.2)

Where the price of the default-free bond is given by;

$$\begin{cases}
    d\mathbb{P}_0(t) = r(t)\mathbb{P}_0(t)dt + \left( \sigma_m^2 + \sigma_c^2 \right) d\mathbb{Z}, & t \in [0, \tau] \\
    \mathbb{P}_0(0) = \mathbb{P}_0 > 0
\end{cases}$$

(3.3)

And equity price is stochastic, risky in both nominal and real terms in the economy and the real price follows a \textit{Itô’s} process that is represented as;

$$\begin{cases}
    d\mathbb{P}_i(t) = \mathbb{P}_i(t) \left\{ u_i(t)dt + \sum_{j=1}^m \sigma_{ij}(t)W^j(t) \right\}, & t \in [0, \tau] \\
    \mathbb{P}_i(0) = \mathbb{P}_i > 0
\end{cases}$$

(3.4)

where $u_i$ is the expected real return on equity per unit of time, $\sigma_{ij} > 0$ is the volatility vector\textsuperscript{12} of the real return on equity per unit of time and $r_i \geq 0$ is the instantaneous spot rate return.

\textsuperscript{12} We assume that the volatility matrix $\begin{bmatrix} \sigma_m & \sigma_c \end{bmatrix}$ has full rank. This assumption ensures that neither the bond nor the stock is a redundant asset in the economy.
The following assumptions are made:

1. The portfolio considered in this paper is assumed to be self financing and continuously rebalanced.

2. Financial Markets are dynamically complete.

3. For the rational investor it is assumed that the value of the expected terminal wealth \( W_n \) satisfies \( \int_{x(1)_d}^{r_s} W_n \geq W_0^{\text{ex}} \).

4. It is assumed that volatility in the credit default swaps of firms is a proxy of credit risk in financial markets.

5. It is assumed that in the familiar Markowitz mean-variance model \( \sigma_p^2 \) is a probability weighted average of market risk \( \sigma_m^2 \) and credit risk \( \sigma_c^2 \).

6. \( W(t) \) is predictable with respects to \( F_{(0)} \) and meets the usual integrating conditions\(^{13}\)

Now let’s consider an agent (investor) with an initial wealth \( W_0 > 0 \) and total wealth over a fixed time interval \( \tau \geq 0 \) of \( W(t) \). Moreover, the agent also receives a stream of investment income \( e(\{e_t\}) \) that he can use to buy additional payoffs of \( n \) different assets at prices \( p \). The investor’s optimization problem can be represented by expressions 1.1 – 1.3 or as;

\[
\max_{u(t) \in U} \left[ u(t) \right] \quad \text{s.t.} \quad \sum_{i=1}^{n} W_i 
\]

(3.5)

where \( u(t) \) = the expected terminal returns of a portfolio of stocks \( s \) and bonds \( b \), and also where \( W(t+1) = e + W \).

Given the agent’s investment strategy \( \{u(t)_0^\infty \} \), asset allocation process \( \{w(t)_0^\infty \} \) that maximizes the expected utility of his wealth and his information set \( \{F_t^B\}_{t \geq 0} \),

\(^{13}\) Harrison-Pliska (1981) and Duffie (1996)
the preferences of the infinitely lived agent can be represented by the time additive Von Newmann-Morgenstern utility function in equation 3.6, with varying levels of risk aversion. Here we follow the Hansen-Richard (1987) and Cochrane (2008) approach which links marginal utility and the mean-variance frontier.

\[
E \left[ \int_0^\infty e^{-\mu t} u(t) \left[ W(t; x, w) \right] dt \right]
\]

(3.6)

We call \( u_i(t), i = 0, 1, 2, \ldots, m \), the total market value of the agent’s wealth in the \( m \) bond \( B_i \) and stock \( S_i \). Where \( u(t) = (u_1(t), \ldots, u_m(t)) \) represents the expected returns on a self financing portfolio which changes over time \( t \). Since \( u(t) \) is self financing it means that;

\[
\frac{dW_i}{W_i} = W_0(t) \frac{dS_i}{S_0} + w'(t)dR(t)
\]

(3.7)

where

\[
dR = \left( \frac{dS_1}{S_1}, \ldots, \frac{dS_N}{S_N}, \frac{dB_r}{B_r} \right)
\]

(3.8)

\( W(t) \) is the portfolio value at \( t \), and \( w'dR \) is a scalar product. The terminal payoff \( W_n \) has finite variance.

**DEFINITION.** (ii) The portfolio strategies are defined by the \( (N+1) \) dimensional vector of weights \( w(t) \) in assets 1, \ldots, \( n \) and the zero-coupon bond \( B_r \). Without loss of generality \( S_{n}(t) \) is defined as reinvested dividends.
From the agent’s utility function in expression 3.6, the agent will try to maximize his expected optimal terminal investment returns \( u^* \) given his decisional wealth constraint;

\[
\begin{align*}
    dW(t) &= \left\{ r(t)W(t) + \sum_{i=1}^{m} (b_i(t) - r(t))u_j(t) \right\} dt + \sum_{j=1}^{m} \left( w_{ij} \sigma_{mij}(t) + w_{2j} \sigma_{cej}(t) \right) u_i(t) dW^j(t) \\
    W(0) &= W_0 \geq l(0), \text{ Otherwise} \quad (3.9)
\end{align*}
\]

given that \( w_1(t) \) denotes the wealth invested in risky assets (stocks), and \( w_2(t) \) is the wealth invested in nominal bonds.\(^{14}\) These weights sometimes have additional constraints,

\[
\psi_i \leq w_i \leq \eta_i, \quad i = 1, 2, \ldots, v
\]

where \( \psi_i \geq 0 \), which represents the “short selling” constraint, is a positive constant, \( \sigma_{mij} \) and \( \sigma_{cej} \) represents the market and credit risk exposure the investor faces, \( u_i(t), i = 0, 1, \ldots, m \) denotes the total market value of the agent’s wealth in the \( i^{th} \) bond or stock and where \( l(0) \) is the lower bound on the decisional variable \( W_0 \geq 0 \). Note that due to the positive lower bound imposed on investments it is clear that wealth should also have a lower positive bound \( \bar{W} \), since the agent needs some minimal amount of wealth for investment which cannot be negative. This condition is expressed in Lemma 1 below.

**Lemma 1.** The process \( W_0 \) describing the agent’s wealth is subject to the following constraint \( W(t) \geq l(t) \quad \forall t > 0 \), where the strictly positive function \( l(t), t > 0 \) represents the solvability level.

Following Hansen and Richard (1987) the study defines \( u(\bullet) \) by its Riesz representation

\(^{14}\) \( w_1 \) and \( w_2 \) are adapted to the information structure \( F_t \). The weights represent the agent’s investment strategy.
in the Hilbert Space as an admissible portfolio strategy\(^\text{15}\) if \(u(\bullet) \in L^2_\mathcal{F}(0, \tau; \mathbb{R}_+^m)\). In fact the pair \((W(\bullet), u(\bullet))\) is an admissible pair if \(W(\bullet) \in L^2_\mathcal{F}(0, \tau; \mathbb{R}_+^m)\) is a solution of the stochastic differential equation in expression 3.9, where the control \(u(\bullet) \in U[0, \tau]\).

4.0 Dynamic Portfolio Problem under Credit Risk

To add clarity to the results this section lays out the framework for the determination and addition of credit risk to the agent’s investment risk exposure. Building the extended dynamic optimization model, the study uses the popular dynamic Markowitz framework as the benchmark, but allows the agent’s investment risk frontier to be a pooled parameter of credit and market risk. Our problem is then to minimize this fulsome risk measure for a given level of return; expression 1.2. The study uses credit default swaps (CDS) as a proxy for credit risk in financial markets. The CDS is a bellwether of increasing (eroding) investor confidence in corporate creditworthiness. A rise (decline) indicates worsening (improving) perceptions of credit quality. Credit default swaps are financial instruments that are used to speculate on a company’s ability to repay debt.

The benchmark model assumes that \(\text{Var}(\sigma_p^2) dZ_t = \sigma_m^2\), where the agent’s risk exposure is only a function of market risk. However as discussed in section 1, since the agent’s credit risk increases over his investment horizon, the extended framework allows credit risk to enter the agent’s investment horizon through a pooled risk parameter. On the other hand the study’s cross-sectional pooled risk framework is assumed to nest the

\(^{15}\) The set of strategies that satisfy the equality and inequality constraints are called the admissible set of the quadratic programming problem.
traditional mean-variance market risk model. In the absence of credit risk, the agent’s wealth constraint depicted in expression 3.9 converges to the benchmark dynamic optimization model in equation 4.0.

\[
\begin{align*}
dW(t) &= \left\{ r(t)W(t) + \sum_{i=1}^{m} (b_i(t) - r(t))u_j(t) \right\} dt + \sum_{j=1}^{m} \sum_{i=1}^{m} (\sigma_{kij}(t))u_i(t) dW^j(t) \\
W(0) &= W_0, \quad Otherwise \\
\end{align*}
\]

(4.0)

Where the general constrained controlled linear stochastic differential notation in 4.0 can be simplified for mathematical ease without loss of generality to notation 4.1 below;

\[
\begin{align*}
dW(t) &= \left\{ A(t)W(t) + B(t)u(t) + f(t) \right\} dt + \sum_{j=1}^{m} D_j(t) u(t) dW^j(t) \\
W(0) &= X_0 - (W_n - u), \quad Otherwise \\
\end{align*}
\]

(4.1)

Where:
- \(A(t)\) and \(f(t)\) are scalars;
- \(u(\cdot) \in L^2_c(0, \tau; \mathbb{R}^m)\);
- \(x(t) = x(t) - (W_n - u)\);
- \(A(t) = r(t)\);
- \(f(t) = (W_n - u)r(t)\);
- \(B(t) = (b_1(t) - r(t), ..., b_m(t) - r(t))\);
- \(D_j(t) = (\sigma_{kij}(t), ..., (\sigma_{Sni}(t) + \sigma_{Cni}(t)))\);
- \(B(t) \in \mathbb{R}^m_+\) and \(D_j(t) \in \mathbb{R}^m (j = 1, ..., m)\) are column vectors.
- The matrix \(\sum_{j=1}^{m} D_j(t)D_j(t)^\prime\) is non Singular.

Following Vasicek (1977), it is assumed that credit risk \(c_t\) (like market risk \(r_t\)) follows an Ornstein-Uhlenbeck diffusion process,\(^{16}\)

\[
dc_t = \kappa(\overline{c}_t - c_t)dt + \sigma_c^2 dz_c
\]

(4.2)

\(^{16}\) Where \(\overline{c}_t\), \(\sigma_c^2\) and \(\kappa\) are positive constants and \(dz_c\) is standard Brownian motion.
where $c$ is the long-run mean, $\sigma_p^2$ is the volatility, and $\kappa$ is the mean reversion.

From assumption 5 in section 3 the investor’s risk frontier is a combination of both market and credit risk, hence the risk frontier is more appropriately modeled as a cross-sectional risk pooled risk frontier which is represented as,

$$VaR\left(\sigma_p^2\right)dz_t = VaR\left(\sigma_m^2 + \sigma_c^2\right)dz_t \quad (4.3)$$

$$\Rightarrow VaR\left(\sigma_p^2\right)dz_t = \frac{(n-1)\sigma_m^2 + (m-1)\sigma_c^2}{n + m - 2} \quad (4.4)$$

where $n$ and $m$ are the number of observations in both sets of risk data.

### 4.1 The Extended Dynamic Cross-Sectional Risk Model

In this section, we derive the extended dynamic efficiency frontier (Cross-sectional Pooled Risk frontier), in the variance-expected return space $\left(\left(\sigma^2_{W_t}\right), E(W_T)\right)$.

$$\Rightarrow \left(\left(\sigma^2_{m}(W_t) + \sigma^2_{c}(W_t)\right), E(W_T)\right)$$

The CSPR frontier consists of payoffs that minimize the portfolio risk which is defined in the following way:

$$\min_{[\mu(\cdot),W(\tau)]} E\left\{\frac{1}{2}W(\tau)^2\right\} = \min_{[\mu(\cdot),W(\tau)]} E\left\{[W_0 - W_\tau]^2 + 2\mu[EW_0 - W_\tau]\right\} \quad (4.1.1)$$

Subject to the constraints in “$R$” solved for different values of the expected terminal wealth ($W_n$)

$$\begin{array}{l}
\left(\begin{array}{l}
E[W_\tau] = W_n \\
\mu(\cdot) \in L^2_{\tau}\left(0, \tau; \mathbb{R}_+^m\right) \\
(W(\cdot), u(\cdot)) \text{satisfies equation 4.0}
\end{array}\right)
\end{array}$$
Given expression 4.1.1 and the controlled linear representation in equation 4.0, our objective is to find a portfolio with the minimum market and credit risk for a given optimal \( u(\tau) \) that minimizes the terminal cost function

\[
J(\tau, w; u(\tau)) = E\left\{ \frac{1}{2} W(\tau)^2 \right\}
\]  

(4.1.2)

Where the associated value function of the controlled linear stochastic differential equation in 4.1 and the stochastic control problem in expression 4.1.2 (the LQ problem) is defined as;

\[
V(\tau, w) = \inf_{u(\tau) \in \mathcal{U}(\tau)} J(\tau, w; u(\tau))
\]  

(4.1.3)

As a necessary condition for optimality, the Hamilton-Jacobi-Bellman (HJB) equation related to the family of stochastic control problems in expression 4.1.3 is given as;

\[
\begin{align*}
&v_t(\tau, w) + \sup_{u \geq 0} \left[ v_w(\tau, w) \left( A(t) w + B(t) u + f(t) \right) + \frac{1}{2} v_{ww}(t, w) u' D(t) \right] D(t) u = 0 \\
v(\tau, w) &= \frac{1}{2} w^2, \quad \text{Otherwise} \\
\end{align*}
\]  

(4.1.4)

Where \( D(t) = \left[ D_1(t), \ldots, D_m(t) \right] \)

The value function \( V(\tau, w) \) defined in equation 4.1.3 is a constrained viscosity solution of the HJB equation in 4.1.4 on the interval \([\tau, +\infty]\). Moreover it follows from the verification theorems of Gozzi and Russo (2006) and the results in Crandall and Lions (1983), Ishii (1987) and Ishii and Loreti (2002) that the HJB as defined by 4.1.4 is an optimal feedback control. In fact using the uniqueness theorems of Crandall and Lions

\[\text{The reader is referred to the optimality conditions for the HJB framework in Fleming and Rishel (1975) for additional insights.}\]
(1983, 1990, 1991) we can show that the value function is a unique smooth viscosity solution of the HJB equation.

Expressed in terms of the investment strategy and adjusted for credit risk, the HJB equation 4.1.4 may be represented as

\[ \text{Sup}_{(u,w) \in \mathcal{A}} \left\{ w_1 [\mu_s - r_s] + w_2 [r_n - \pi] + W_t \right\} \frac{dd}{dw} + u(\cdot) + \frac{1}{2} \left[ w_1 \left( \sigma_s^2 + \sigma_n^2 \right) - w_2 \left( \sigma_x^2 \right) \right] \frac{d^2d}{dw^2} = \rho d(w) \quad (4.1.5) \]

Where \( \rho > 0 \) is a unique strict global “minimizer” or constant discount factor, and

\[ H(x, Dv(x), D^2v(x)) \triangleq \sup_{\theta \in [0,1]} \left\{ U(x) + \left[ (\theta \sigma \lambda + r) x + \alpha w - \eta \alpha w e^{\theta - 1} \right] D_v(x) + \frac{1}{2} \theta^2 \sigma^2 x^2 D^2v(x) \right\} \quad x \in [l, +\infty) \]

is the generalized Hamiltonian.

**DEFINITION.** (iii) A vector \( \rho^* \in \mathbb{R}^m \) is called a strict global minimizer of expression 4.1.6, if \( \rho^* \in F \) and there is a neighborhood \( U(\rho^*) \in \mathbb{R}^m \) such that \( f(\rho) \geq f(\rho^*) \) for all \( \rho \in u(\rho^*) \cap F \).

To find \( \rho \) we put forward the following Lemma;

**LEMMA 2.** Let \( Q \) be a continuous, strictly convex quadratic function

\[ Q(\rho) \triangleq \frac{1}{2} \left\| (D')^{-1} \rho + (D')^{-1} B' \right\|^2 \quad (4.1.6) \]

Over \( \rho \in [0, \infty)^m \), where \( B' \in \mathbb{R}^m \), \( D \in \mathbb{R}^{m \times m} \) and \( DD' > 0 \). Then \( Q \) has a unique strict global “minimizer” or constant discount factor \( \rho \in [0, \infty)^m \).

Hence Lemma 2 implies that;

\[ \left\| (D')^{-1} \rho + (D')^{-1} B' \right\|^2 \leq \left\| (D')^{-1} \rho (D')^{-1} B' \right\|^2 \quad \forall \rho \in [0, \infty)^m \quad (4.1.7) \]

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The Kuhn-Tucker condition for minimization of $Q$ in expression 4.1.6 over $[0, \infty)^m$ lead to the Lagrange multiplier vector\(^\text{18}\) $\bar{\theta} \in [0, \infty)^m$ such that

$$\bar{\theta} = \nabla Q(\bar{\rho}) = \left[ (D'D)^{-1} \bar{\rho} + (D'D)^{-1} B' \right]$$  \hspace{1cm} (4.1.8)

where $\bar{\theta} \bar{\rho} = 0$ and $\bar{\xi} = (D')^{-1} \bar{\rho} + (D')^{-1} B'$ (The Lagrange multiplier vector)

$$\Rightarrow \bar{\theta} = \nabla Q(\bar{\rho}) = D^{-1} \bar{\xi}$$

$$\Rightarrow \bar{\rho} D^{-1} \bar{\xi} = 0$$

$$\Rightarrow Q(\alpha \bar{\theta}) = Q(\alpha D^{-1} \bar{\xi}) = \frac{1}{2} \alpha^2 \| \bar{\xi} \|^2$$

So given this unique minimizer\(^\text{19}\) derived in expression 4.1.6 and the Lagrange multiplier in 4.1.8, the global minimizer and the Lagrange multiplier for the extended dynamic optimization model can be expressed as;

$$\bar{\rho}(\tau) = \arg \min_{\rho(\tau) \in [D, \infty)^m} \frac{1}{2} \| (\sigma^2_M + \sigma^2_C)(t)^{-1} \rho + (\sigma^2_M + \sigma^2_C)(t)^{-1} (b(t) - r(t) - \pi(t))1 \|^2$$  \hspace{1cm} (4.1.9)

and;

$$\bar{\theta}(t) \doteq (\sigma^2_M + \sigma^2_C)(t)^{-1} \bar{\rho}(t) + (\sigma^2_M + \sigma^2_C)(t)^{-1} (B(t) - r(t) - \pi(t))1$$  \hspace{1cm} (4.1.10)

Hence from equations 4.1.9 and 4.1.10 the optimal portfolio selection strategy in the presence of both market and credit risk that corresponds to the expected terminal wealth $E[W(\tau)] = W_n$, as a function of time $t$ and initial wealth $W_0$ can be expressed as;

$$u^*(t,W) \equiv [u_1^*(t,W), ..., u_m^*(t,W)]$$

\(^{18}\) The Lagrange Multiplier Vector ($\theta$) approximates the marginal impact on the objective function caused by a 1 unit change in the constant of the constraint.

\(^{19}\) This unique global minimizer may also be referred to elsewhere in the optimization literature as a constant discount factor.
\[
\begin{align*}
\mu^* &= \frac{d - W_0 e^\gamma}{1 - e^\gamma} \\
\end{align*}
\]

Thus the agent’s efficiency frontier is represented as:

\[
\begin{align*}
\text{Var}[W(\tau)] &= \frac{\left( d - W_0 e^\gamma \right)^2}{\frac{\int [\overline{\rho}(s)]^2 ds}{e^\gamma - 1}} \equiv \frac{\left( E[W(\tau)] - W_0 e^\gamma \right)^2}{\frac{\int [\overline{\rho}(s)]^2 ds}{e^\gamma - 1}} \\
\end{align*}
\]

**LEMMA 2.** The following relation holds for \( \int_t^\tau r(s)ds \) and \( \int_t^\tau \| \overline{\theta} \|^2 ds \) over \( \tau \in [0, \infty] \)

\[
\int_t^\tau r(s)ds = \gamma(t)(\tau - t) - \int_t^\tau \phi_{\gamma(t)}(s)dz_\gamma(s)
\]

When \( t=0 \)

\[
\Rightarrow E\int_0^\tau r(t)dt = \gamma\tau
\]

hence \( \int_t^\tau r(s)ds \sim N[\gamma(t)(\tau - t)] \)
5.0 Data and Empirical Illustration

In this Section, we demonstrate that the extended dynamic analysis of Section 4 can easily be adapted to alternative economic environments. Section 5.1 illustrates the analytical flexibility of our methodology in relation to a benchmark baseline dynamic framework, and provides an explicit solution to the extended dynamic credit risk model.

The study selected 10 actively traded stocks, 2 corporate bonds and the U.S. 10-year Treasury bond to illustrate the approach for estimating the extended dynamic optimization model. The corporate bonds are General Electric’s (GE) CUSIP# 369604AY9 and JPMorgan’s (JPM) CUSIP# 014037179, while the stocks are Alcoa (AA), Procter & Gamble (PG), McDonald’s (MCD), Disney (DIS), Wal-Mart (WMT), American Express (AXP), AT&T (T), Boeing (BA), Caterpillar (CAT) and International Business Machines (IBM). The study chose representative stocks from the Dow Jones industrial-30 covering a variety of industries and which had high trading volumes on the NYSE. Trade data for these stocks and bonds were taken from Bloomberg and covered 7 years, ranging from July 31st 2001 to July 31st 2008. Days with no trading activities were eliminated from the study. From this data, the return of each individual stock and corporate bond \( r_t \), was calculated for each month so the average monthly return was given by

\[
\bar{r} = \frac{\sum_{t=1}^{n} r_t}{N},
\]

where \( N \) is the number of observations. Using this data, the covariance between each pair of stocks is then calculated by

\[
\text{cov}(r_a, r_b) = \frac{\sum_{t=1}^{n} (r_{a,t} - \bar{r}_a)(r_{b,t} - \bar{r}_b)}{N-1}
\]
For ease of computation but without loss of generality we assume that $w_i$ is evenly weighted across all asset classes in the portfolio.

Table 1: Summary Statistics of the study’s portfolio’s risk and returns used to obtain the optimal solutions for the two portfolio strategies

<table>
<thead>
<tr>
<th>Firm Name</th>
<th>Ticker</th>
<th>Industry</th>
<th>CR</th>
<th>$\mu$</th>
<th>$\sigma_r^2$</th>
<th>$\sigma_c^2$</th>
<th>$\sigma_p^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 Alcoa</td>
<td>AA</td>
<td>Aluminium</td>
<td>BBB</td>
<td>2.89</td>
<td>8.94%</td>
<td>21.78%</td>
<td>15.36%</td>
</tr>
<tr>
<td>2 American Express</td>
<td>AXP</td>
<td>Consumer Finance</td>
<td>A</td>
<td>2.36</td>
<td>5.81%</td>
<td>29.02%</td>
<td>17.42%</td>
</tr>
<tr>
<td>3 AT&amp;T</td>
<td>T</td>
<td>Telecommunications</td>
<td>BBB</td>
<td>1.33</td>
<td>7.33%</td>
<td>23.25%</td>
<td>15.29%</td>
</tr>
<tr>
<td>4 Boeing</td>
<td>BA</td>
<td>AeroSpace &amp; Defence</td>
<td>BBB</td>
<td>5.50</td>
<td>7.31%</td>
<td>23.54%</td>
<td>15.43%</td>
</tr>
<tr>
<td>5 Caterpillar</td>
<td>CAT</td>
<td>Commercial Vehicles</td>
<td>A</td>
<td>24.59</td>
<td>7.20%</td>
<td>17.37%</td>
<td>12.29%</td>
</tr>
<tr>
<td>6 General Electric</td>
<td>GE</td>
<td>Industrial</td>
<td>AAA</td>
<td>1.72</td>
<td>5.44%</td>
<td>24.98%</td>
<td>15.21%</td>
</tr>
<tr>
<td>7 International Business Machines</td>
<td>IBM</td>
<td>Computer Services</td>
<td>A</td>
<td>9.53</td>
<td>7.20%</td>
<td>20.49%</td>
<td>13.85%</td>
</tr>
<tr>
<td>8 JPMorgan Chase Bank</td>
<td>JPM</td>
<td>Banking</td>
<td>A</td>
<td>7.31</td>
<td>8.42%</td>
<td>24.94%</td>
<td>16.68%</td>
</tr>
<tr>
<td>9 McDonalds</td>
<td>MCD</td>
<td>Restaurants</td>
<td>BBB</td>
<td>16.49</td>
<td>6.92%</td>
<td>21.81%</td>
<td>14.37%</td>
</tr>
<tr>
<td>10 Procter &amp; Gamble</td>
<td>PG</td>
<td>Consumer Products</td>
<td>A</td>
<td>5.49</td>
<td>4.03%</td>
<td>15.21%</td>
<td>9.62%</td>
</tr>
<tr>
<td>11 WalMart</td>
<td>WMT</td>
<td>Retailer</td>
<td>A</td>
<td>2.24</td>
<td>5.00%</td>
<td>21.43%</td>
<td>13.22%</td>
</tr>
<tr>
<td>12 Disney</td>
<td>DIS</td>
<td>Consumer Entertainment</td>
<td>BB</td>
<td>8.56</td>
<td>6.22%</td>
<td>22.22%</td>
<td>14.22%</td>
</tr>
</tbody>
</table>

Notes:
- $\mu$ - Mean Returns
- $\sigma^2$ - Variance
- CR - Credit Rating
- *GE Bond - CUSIP
- *JPMorgan Bond - CUSIP

5.1 Discussion

From table 1, the study’s portfolio was comprised of 10 equity products and 2 corporate bonds. Here we assume that the asset allocation was along the lines of approximately 80 percent equity and 20 percent fixed income. From the portfolio mix we let $m = 12$, whilst the interest rate on the 10-Year U.S. Treasury bond obtained from Bloomberg at the close of trading on July 31st 2008 was 4.02% and the appreciation rate of the $m$ stocks $= (x_1, x_2, ... x_n)'$. The resulting LaGrange Multiplier and unique minimizer in equations 4.1.9 and 4.1.10 are derived as follows;

$$\bar{\theta}(t) = (x_1, x_2, ... x_n)'$$
while the unique minimizer over \([0, \infty)^m\) is given as

\[ \bar{\rho}(\tau) = [4.02, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T \]

with a minimum value\(^{20}\)

\[ s(\bar{\rho}) = \|\sigma^{-1}\bar{\rho} + \theta\|^2 = 0.0403 \]

for the benchmark model and 0.0092 for the extended model.

Hence from Lemma 3 the agent’s efficient portfolio (allocation) strategy under the benchmark framework would be;

\[
\begin{align*}
 u^*(t, W) = & \begin{cases} 
 0.00 \\
 0.00 \\
 0.77 \\
 1.53 \\
 1.31 \\
 0.76 \\
 1.14 \\
 0.00 \\
 0.00 \\
 0.00 \\
 0.00 \\
 0.00 \\
 0.00 \\
 0.00 \\
 0.00 \\
 0.00
\end{cases} & \text{if } W_0 - (W_n - \mu^*)e^{4.02(\tau-t)} \leq 0 \\
 & \text{if } W_0 - (W_n - \mu^*)e^{4.02(\tau-t)} > 0 
\end{align*}
\]

(5.1.1)

Where the optimal strategy attains its maximum value at

\[
\mu^* = \frac{d - W_0 e^{\tau}}{1 - e^{0.0403\tau}} = \frac{d - W_0 e^{4.02\tau}}{1 - e^{0.0403\tau}}
\]

(5.1.2)

\(^{20}\) Suppose \(u\) and \(v\) are two orthogonal vectors in \(\mathbb{R}^n\) then, \(\|u + v\|^2 = \|u\|^2 + \|v\|^2\).

**Proof:** The proof of this theorem is fairly simple. From the proof of the triangle inequality for norms we have the following statement.

\[
\|u + v\|^2 = \|u\|^2 + 2(u + v) + \|v\|^2
\]

However because \(u\) and \(v\) are orthogonal we have \(u^*v = 0\) and so we get; \(\|u + v\|^2 = \|u\|^2 + \|v\|^2 \)

\(\square\)
And his efficient frontier can be written as:

\[
Var[W(\tau)] = \left( \frac{d - W_0 e^{\theta \tau}}{\int_{\tau}^{\tau} e^{\theta s} ds} - 1 \right) (E[W(\tau)] - W_0 e^{4.02 \tau})^2
\]

(5.1.3)

For the extended model we find that the expected terminal return is given by

\[
\mu^* = \frac{d - W_0 e^{\theta \tau}}{1 - e^{\theta \tau}} = \frac{d - W_0 e^{4.02 \tau}}{1 - e^{0.0402 \tau}}
\]

(5.1.4)

Then the efficient frontier is represented as

\[
Var[W(\tau)]_{Traditional} > Var[W(\tau)]_{Extended}
\]
First, the study demonstrates in equations 5.1.2 through 5.1.5 that given credit risk
the investor’s true optimal dynamic asset allocation is lower than previously indicated by
the benchmark dynamic framework. In fact the investor’s true terminal return may lie on
or between the upper market-risk boundary and the lower credit-risk enhanced boundary
as graphically illustrated using sample data in figure 3. In addition the graph shows that
in the presence of credit risk, the investor’s efficiency frontier moves inwards to a lower
terminal return as illustrated by equations 5.1.2 and 5.1.3 respectively. However since the
extended model is nested in the benchmark dynamic framework, when credit risk
dissipates $c_r \leq 0$ then the agent’s risk frontier and terminal return converges to that of the
benchmark model.

Similarly, the analysis shows that the agent’s efficient strategy of portfolio
selection corresponding to the expected terminal wealth differs under both risk
measurement scenarios. Expression 5.1.1 indicates that given the level of optimal return
obtained from equations 5.1.2 and 5.1.4, the strategy that works for the market-risk only
scenario would not work in an investment environment involving both market and credit
risk. Hence as indicated in this analysis this agent will modify his investment strategy so
as to better adapt to the alternative credit risky investment environment.

\[
\text{Var}[W(\tau)] = \left( d - W_0 e^h \int_{\tau}^{\tau} r(s) ds \right)^2 \frac{\int_{\tau}^{\tau} e^{\sigma(s)} ds}{e^0 - 1} \equiv \left( \frac{E[W(\tau)] - W_0 e^{4.02\tau}}{e^{0.0092\tau} - 1} \right)^2
\]
Secondly, the closed form solution in expression 5.1.8 demonstrates that given credit risk, investors exhibit a greater level of risk aversion $\left( \frac{1}{\phi_E} \right)$ as indicated by a larger risk aversion coefficient. Notice that since 5.1.5 has the smaller $u^*$ it follows that he will have a greater risk aversion coefficient when the inverse is taken in expression 5.1.8.

$$u^* = u^f e^0 \left( \sigma_p^2, \phi(*) \right) + \frac{1}{\phi} \left( \bar{u} - u^f e^0 \left( \sigma_p^2, \phi(*) \right) \right) = \frac{d - W_0 e^0}{1 - e^0}$$

(5.1.6)

$$\Rightarrow 1 = \frac{u^* - u^f e^0}{\bar{u} - u^f e^0}$$

(5.1.7)

$$\Rightarrow \varphi = \left[ \frac{u^* - u^f e^0}{\bar{u} - u^f e^0} \right]^{-1}$$

(5.1.8)
\[ \Rightarrow \frac{1}{\varphi_e} > \frac{1}{\varphi_B} \]

Finally, the study shows that the added (credit) risk results in a reduction in the agent’s terminal return on his investment because of the investor’s level of risk aversion. In fact, the analysis adds support to the longstanding view that during periods of increased credit risks, investors reduce their holdings of credit risky products and move to the safety of the lower yielding risk-free products such as U.S. Treasury instruments, which results in an overall lower terminal portfolio return.

6.0 Conclusion

Bajeux and Portait (1998) concluded from their study on dynamic asset allocation that the dynamic efficient framework outperformed the standard static framework. Moving a step further, this study demonstrates that the credit risk enhance framework is fundamentally much more flexible and dynamic than the traditional dynamic framework. The paper first established a baseline dynamic optimization model which was used to determine an optimal terminal return given market risk. A more fulsome risk model inclusive of credit risk was later developed to investigate investors’ attitude to credit risk. The empirical illustration of the extended cross-sectional pooled risk model demonstrates that the dynamic optimal portfolio return is lower than indicated by the benchmark Markowitz mean-variance framework because traditional models implicitly assumes the non-existence of credit events.

The inclusion of credit risk shows that because of investors’ aversion to credit risk the investor’s true optimum is below the benchmark optimum or within a given boundary region as illustrated in figure 2. Moreover, the extended model exhibits the analytical
flexibility whereby changes in credit risk reflect investors’ decisions as they move to minimize portfolio risks for a given level of return. In fact the model demonstrates the “flight-to-quality” phenomenon in the presence of credit risks which is characterized by generally lower level of portfolio return.

From the standpoint of policy this work will not only compliment past empirical work in dynamic asset allocation but will also provide investors a vehicle for determining a more fulsome measure of perceived portfolio risk in an investment environment characterized by deteriorating (improving) credit quality and rising (falling) market risk (interest rates). The flexibility of the model is highlighted in the fact that in the absence of credit risks, the model converges to the standard dynamic optimization framework.
REFERENCES


