A RATIONALIZATION OF THE PRECAUTIONARY DEMAND FOR CASH*

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INTRODUCTION

Application of the principles of inventory theory to the transactions demand for cash 1 and use of the strategy of portfolio selection to examine the demand to hold money as an asset 2 have represented significant responses to Hicks's suggestion that the theory of rational choice be incorporated into the demand for money. 3 In this article an attempt is made to apply the logic of rational behavior to another of the Keynesian triad of motives for holding cash: the precautionary demand for cash.

First, the precautionary demand for cash will be defined, and then within the confines of this definition the concepts needed to determine optimal precautionary cash balances will be described. After establishing an equation for optimal precautionary cash balances and examining its characteristics, this approach to the precautionary demand for cash will be compared with the inventory approach to the transactions demand for cash and with related studies.

DEFINITION OF THE PRECAUTIONARY DEMAND FOR CASH

According to Keynes precautionary cash balances are those which are held "to provide for contingencies requiring sudden ex-

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penditure and for unforeseen opportunities of advantageous purchases, and also to hold an asset of which the value is fixed in terms of money to meet a subsequent liability in terms of money. . . ." 4 This definition requires an extension and qualification.

It should be added that not only do precautionary balances apply to fortuitous expenditures but also to the unpredictability of receipts. Financial embarrassment may result from either unexpected additional disbursements or the failure of expected receipts to be realized. The qualification to Keynes's definition is with regard to the third aspect of the precautionary motive. Holding cash to meet a subsequent liability is appropriate to the precautionary motive if, as is the case with demand deposits of banks, the time of repayment of the liability is unpredictable. If, however, the maturity or repayment date of the liability is determined with certainty, the cash holding designed to meet this payment of debt more appropriately falls within the province of transactions cash balances.5

When a firm's pattern of receipts and disbursements is not known with certainty, disbursements may exceed receipts during a given period by an unpredictable amount. In this situation, therefore, it is required or advisable that a firm retain a certain amount of cash to meet the possibility of an excess of disbursements over receipts. The problem is to determine the optimal quantity of precautionary cash balances.

THE CONCEPTS

Three factors affect the optimal size of precautionary cash balances: (1) the cost of illiquidity; (2) the opportunity cost of holding precautionary cash balances; and (3) the average volume and variability of receipts and disbursements. The cost of illiquidity refers to the seriousness of the consequences of underestimating cash needs for a given payments period. If a firm has no suitable collateral or available lines of credit, failure to provide adequately for necessary disbursements may result in insolvency and declaration of bankruptcy. Presumably the cost of illiquidity is quite high in this case. If credit is available, the cost of illiquidity then will

5. Transactions cash balances, therefore, are confined to those balances held to meet future obligations known with certainty, and uncertainty is reflected entirely in precautionary cash balances. Although empirically this distinction may seem artificial, analytically it follows from Keynes's definition of precautionary cash balances and the Baumol-Tobin model for the transactions demand for cash.
"depend on the cheapness and reliability of methods of obtaining cash, when it is required, by some form of temporary borrowing."  

6 If an individual or firm possesses assets which are readily convertible to cash, the cost of illiquidity will be the implicit and explicit cost of a transfer transaction making this conversion. In this third case, the cost of illiquidity corresponds closely to the "brokerage" costs discussed by Baumol in connection with transactions cash balances.  

7 Since cash earns no explicit return, a firm incurs an opportunity cost when it holds cash instead of alternative income-earning assets. The opportunity cost of holding precautionary cash balances is very similar to the opportunity cost of holding transactions cash balances.  

8 As precautionary cash balances increase, opportunity costs increase. 

The variability and average volume of receipts and disbursements also influences the size of precautionary cash balances. For a given time period, a firm has an expected volume of receipts and disbursements. Expected disbursements minus expected receipts defines a probability distribution of expected net disbursements. The probability distribution of expected net disbursements is assumed to have a mean of zero and a standard deviation which is determined by the degree of uncertainty attached to the pattern of expected receipts and disbursements and by the average volume of receipts and disbursements. As the average volume of receipts and disbursements increases, the standard deviation of the distribution of expected net disbursements will tend to increase because possible discrepancies between receipts and disbursements increase. The exact nature of the relationship between the average volume of receipts and disbursements and the standard deviation of net disbursements depends on the type of probability distribution which is assumed. Whatever the cause, an increase in the standard deviation of the net-disbursements distribution will require an increase in precautionary cash balances in order to maintain a given probability against insufficient cash-on-hand.

**Optimal Precautionary Cash Balances**

The total cost of precautionary cash management is equal to the sum of the opportunity cost of holding cash and the expected cost of illiquidity. The opportunity cost is determined by multiply-

ing average precautionary cash balances by the rate of return on income-earning assets which are alternatives to cash; that is:

\[ M \cdot r \]

stands for the opportunity cost, where \( M \) is average precautionary cash balances, and \( r \) is the opportunity cost rate.

The expected cost of illiquidity, on the average, is equal to the cost of failing to provide adequate cash funds to meet required disbursements times the probability of this type of financial embarrassment occurring. Therefore, the expected cost of illiquidity can be shown as:

\[ p \cdot c, \]

where \( p \) stands for the probability that net disbursements will be greater than precautionary cash balances, and \( c \) represents the cost per occurrence of insufficient cash. For simplicity, the size of a cash deficiency is assumed not to affect the cost of illiquidity. In other words, this model deals only with the constant costs of illiquidity; a more complete description incorporating both constant costs and those which vary with the size of a cash deficiency is presented in the Appendix.

Using these symbols, an equation for the total cost of precautionary cash management can be constructed:

\[ (1) \quad E = M \cdot r + p \cdot c, \]

where \( E \) stands for the total cost of precautionary cash management. This equation can be simplified by considering the relationship between the size of precautionary cash balances and the probability that net disbursements will exceed this reserve.

From Tchebycheff's inequality a function which conservatively but generally relates the probability of insufficient cash to precautionary cash balances can be constructed. According to this theorem, the probability that a variable will deviate from its mean by more than \( k \) times its standard deviation is equal to or less than \((1/k^2)\).\(^9\) Since the probability distribution of net disbursements is

9. Acheson J. Duncan, *Quality Control and Industrial Statistics* (Homewood, Ill.: Irwin, 1959), p. 69. Less conservative estimates of the probability that an observation will deviate from the mean by more than some multiple of the standard deviation can be used. If, for example, it is assumed that the net-disbursements distribution is unimodal and that the difference between the mode and the mean does not exceed the standard deviation, the Camp-Meidell inequality shows that the probability that a variable will deviate from its mean by more than \( k \) times its standard deviation is equal to or less than \((1/2.25k^2)\). If it is further assumed that the distribution is symmetrical about the mean, one-half the probability determined by the Camp-Meidell inequality will be appropriate, since only those values in which disbursements exceed receipts necessitate precautionary cash balances. However, since
assumed to have an expected value of zero, the multiple of the standard of deviation, $k$ can be expressed as:

$$k = \frac{M}{s},$$

where $s$ is the standard deviation of the distribution of net disbursements. Therefore, the probability that net disbursements will exceed precautionary cash balances can be shown in the following inequality:

$$P \leq \frac{1}{(M/s)^2}.$$

If the most conservative estimate of the probability that precautionary cash balances will be exceeded is assumed to be the relevant estimate for precautionary cash management, the inequality becomes, after rearranging terms:

$$p = \frac{s^2}{M^2}.$$

When the above expression for $p$ is substituted into equation (1), the total cost of precautionary cash management becomes:

$$E = M \cdot r + \left(\frac{s^2}{M^2}\right)c.$$

As precautionary cash balances increase, the expected cost of illiquidity tends to diminish; however, the opportunity cost of holding precautionary cash balances increases. The objective of a profit-maximizing firm is to choose that amount of precautionary cash balances which will minimize the sum of these two costs. The total cost of precautionary cash management will be minimized by increasing precautionary cash balances up to the point at which the marginal increase in opportunity cost becomes equal to the marginal decrease in the expected cost of illiquidity.\(^1\)

An equation for precautionary cash balances that will minimize the total cost of precautionary cash management can be determined by taking the derivative of equation (2) with respect to $M$, setting the derivative equal to zero, and solving for $M$. The equation which results is:

$$M = \sqrt[3]{\frac{2s^2c}{r}}.$$

This equation shows that optimal precautionary cash balances will vary in proportion with the cube root of (1) the variance of the Tchebycheff's inequality is the simplest and the most general, and since use of the other inequalities does not affect the reasoning of the analysis, it has been adopted in the model which follows. A model using the narrower limits of the normal distribution is presented in the Appendix.

1. This relationship was suggested in an article by T. M. Whitin, "Inventory Control in Theory and Practice," this Journal, LXVI (Nov. 1952), 505–8.
distribution of net disbursements, (2) the cost of illiquidity, and (3) the reciprocal of the opportunity cost rate.

At this stage of the analysis, the relationship between the variance, $s^2$, of the net-disbursements distribution and the average volume of receipts and disbursements becomes particularly interesting. As was noted earlier, the exact nature of this relationship depends on the type of probability distribution of net disbursements which is assumed. If the average amount of each receipt and disbursement remains the same, but if the number of receipt-and-disbursement transactions varies, the standard deviation of a normal net-disbursements distribution will vary with the square root of the volume of receipts and disbursements. In this case, then, the variance of the distribution and the volume of receipts and disbursements will vary proportionately. Therefore, according to the equation for optimal precautionary cash balances, when a normal distribution of net disbursements is assumed, optimal precautionary cash balances change in proportion with the cube root of changes in the average volume of receipts and disbursements.

For other distributions, the variance and the average volume of receipts and disbursements will not necessarily change proportionately. However, as indicated by the simple model in equation (3), as long as the variance changes less than in proportion with the cube of the average volume of receipts and disbursements, precautionary cash balances will vary less than in proportion with changes in the volume of receipts and disbursements.

It is pointed out by Patinkin, using a somewhat different approach to the investigation of precautionary cash balances, that the precautionary cash balance required to maintain a given probability against financial embarrassment will increase less than in proportion with an increase in the volume of receipts and disbursements if this increase "is generated by an increase, not in the average value of each of . . . [a firm's] contracts, but in their number." An in-

2. The square-root relationship between the size of a sample and its standard deviation is noted by William J. Baumol, op. cit., p. 556. As Baumol points out, the first application of the square-root relationship was to bank reserves by F. Y. Edgeworth, "The Mathematical Theory of Banking," Journal of the Royal Statistical Society, LI (1888), 123-27. In Kenneth J. Arrow, Samuel Karlin, and Herbert Scarf, Studies in the Mathematical Theory of Inventory and Production (Stanford University Press, 1958), p. 7, Arrow suggests that the probability of an occurrence of insufficient cash should not be considered given but should be related to opportunity costs and costs of illiquidity — as has been done in equation (3).


crease in the average size of each receipt or disbursement will necessitate a proportionate increase in precautionary cash balances in order to maintain the same level of probability against insufficient funds.

Equation (3) concurs with this conclusion. Given the other variables affecting the demand for precautionary cash balances, a doubling of the price level, for example, will result in a doubling of the average value of each transaction by a firm, but the number of transactions will remain the same. The average volume of receipts and disbursements will double, and the variance of the net-disbursements distribution will vary with the square of the volume of receipts and disbursements, or quadruple. The combined effect of the quadrupling of the variance and doubling of the cost of illiquidity — since all prices, including this one, have doubled — is equal to the cube root of eight, which is two. Therefore, the demand for precautionary cash balances varies in proportion with changes in the price level.

Comparison with Transactions Demand for Cash

The preceding analysis of the precautionary demand for cash developed according to optimizing criteria indicates a striking similarity between the precautionary demand for cash and the inventory approach to the transactions demand for cash. The opportunity cost of holding precautionary cash balances and the cost of illiquidity have their counterparts in the opportunity cost of holding transactions cash balances and the cost of transfer transactions between cash and income-earning assets. Under simplifying assumptions, respective to the analysis of each, the relationships between these two sets of costs and the precautionary and transactions demand for cash are similar. As opportunity costs increase and as costs of transfer and illiquidity decline, the demand for both of these cash balances tends to diminish. In addition, under moderately restrictive assumptions, it can be demonstrated that the optimal values of both precautionary and transactions cash balances will tend to vary with the cube and square root, respectively, of the volume of receipts and disbursements.

Comparison with Related Studies

Since its introduction, the precautionary demand for cash has been variously interpreted. A brief review of these interpretations
seems appropriate in order to ascertain to what extent the approach developed above deviates from them.

In his definition of the precautionary motive, Keynes indicates that the demand for precautionary cash balances is sensitive to the opportunity cost of holding cash, an asset which earns no explicit return, and also that the magnitude of the demand is affected by the seriousness of the consequences of lacking sufficient funds to meet necessary disbursements.\(^5\) Later, however, in his discussion of the nature of the liquidity function, he treats the behavior of those cash balances held for transactions and precautionary motives as being similar and characterizes them as being insensitive to the interest rate, which is one of the opportunity costs of holding cash.\(^6\) In addition, he postulates that, in the short run, velocity will be constant, inferring that precautionary cash balances—as well as transactions cash balances—will vary proportionately with changes in the volume of expenditures.

Subsequent writers frequently have associated the precautionary demand for money more closely with the speculative than with the transactions demand for money. Alvin Hansen divides the liquidity function into two analytical categories: (1) the transactions-demand function, and (2) the asset-demand-for-money function.\(^7\) He includes in this latter component the speculative and precautionary demands for money and characterizes it as being sensitive to interest rates in the short run, but relatively insensitive to changes in the volume of income and expenditure.\(^8\)

Other writers have followed this association. "As uncertainty grows, speculators want a larger proportion of riskless cash in their portfolios. At this point the speculative and margin-of-safety [precautionary] motives converge."\(^9\) More recently, Harry Johnson has taken the position that Keynes's final formulation of the speculative motive includes the precautionary motive,\(^1\) and that attempts to state the speculative or liquidity-preference motive for holding cash in terms of uncertainty about future interest rates "is really the precautionary motive in disguise."\(^2\)

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In order to place in perspective the model for the precautionary demand for cash which has been developed in this paper, it may be instructive to conclude by comparing it with these interpretations. Although the precautionary and speculative motives for holding cash both deal with uncertainty, the model for the precautionary demand for cash relates the demand for cash to uncertainties regarding the pattern of receipts and disbursements; the speculative motive, on the other hand, is more directly concerned with uncertainties regarding interest rates. Therefore, in accordance with the association adopted by Keynes, transactions and precautionary cash balances appear to be more closely related than are precautionary and speculative cash balances. Nevertheless, it also appears that precautionary and transactions cash balances possess greater interest elasticity than he ascribed to them in his discussion of the liquidity function.

**APPENDIX**

**CONSTANT-COST MODEL WITH NORMAL NET DISBURSEMENTS DISTRIBUTION**

In this model for the precautionary demand for cash, expected net disbursements are assumed to be normally distributed. When the expected cost which varies with the size of a cash deficiency is excluded from consideration, the total cost of precautionary cash balance management is equal to the sum of (1) the opportunity cost of holding cash, and (2) the expected cost of an incident of illiquidity.

The opportunity cost is defined, as before, as:

\[ M \cdot r, \]

where \( M \) is the amount of precautionary cash balances, and \( r \) is the rate of return on income-earning assets which are alternatives to cash. The expected cost of an incident of illiquidity is defined as the cost of an incident of illiquidity, \( c \), times the probability that net disbursements will be greater than precautionary cash balances. If the probability function of net disbursements, \( D \), can be described by the frequency function \( f(D) \), then the expected cost of an incident of illiquidity is:

\[ c \cdot \int_{M}^{\infty} f(D) dD. \]

The equation for the expected cost of precautionary cash balance management, therefore, takes the general form:

\[(A1) \quad E = M \cdot r + c \cdot \int_{M}^{\infty} f(D) dD.\]

Taking the first derivative of this equation with respect to \( M \), setting
the derivative equal to zero, substituting for \( f(D) \) the frequency function for a normal distribution, and solving for \( M \) results in the following expression for optimal precautionary cash balances:

\[
M = \bar{D} + s \sqrt{\frac{2 \log_e (c/r \cdot s \cdot \sqrt{2\pi})}{2}},
\]

where \( \bar{D} \) and \( s \) are the mean and standard deviation of the distribution, respectively. This equation maintains the conclusion established by the general model in the text above that optimal precautionary cash balances will vary positively and less than in proportion with changes in the average volume of receipts and disbursements, the cost of an incident of illiquidity, and the reciprocal of the opportunity cost rate.

**Model with Constant and Proportional Costs of Illiquidity**

Extension of the model to include a consideration of the expected cost which varies with the size of a cash deficiency can be accomplished only by placing restrictions on the frequency function for net disbursements. The expected value of the cost which varies with the size of a cash deficiency is defined as the sum of the expected values of all possible cash deficiencies times the cost per dollar of a cash deficiency, \( v \). If the expected value of a cash deficiency is defined as the size of a cash deficiency times its probability of occurrence, then:

\[
\int_{M}^{\infty} (D - M) f(D) dD
\]

represents the sum of the expected values of all possible cash deficiencies, and

\[
v \cdot \int_{M}^{\infty} (D - M) f(D) dD
\]

represents the expected value of the cost which varies with the size of a cash deficiency.

The equation for the cost of precautionary cash balance management, therefore, takes the general form:

\[
E = M \cdot r + c \cdot \int_{M}^{\infty} f(D) dD + v \cdot \int_{M}^{\infty} (D - M) f(D) dD.
\]

The first derivative of this equation taken with respect to \( M \) and set equal to zero is:

\[
E' = r - c \cdot f(M) - v \int_{M}^{\infty} f(D) dD = 0.
\]

In order to derive from equation (A4) an expression for optimal precautionary cash balances, it must be possible (1) to evaluate the integral of the frequency function for an explicit expression, and (2) to solve the first derivative set equal to zero for \( M \). The first requirement precludes use of the normal distribution, and the second suggests that a simple frequency distribution will expedite finding a solution for \( M \).

Two of the simplest frequency distributions are the rectangular and triangular distributions. Both require that the limits be spec-
ified, so expected payments are assumed to represent the upper bound of the net disbursements distribution, and expected receipts are assumed to represent the lower. The implication of this assumption is that the largest possible cash outflow is a situation in which only disbursements occur; the greatest possible cash inflow is one in which no disbursements are necessary but in which expected receipts are fully realized. It is further assumed that expected disbursements are equal to expected receipts, so that the expected value of the net disbursements distribution is zero.

Although the rectangular distribution presumably does not provide a realistic form for empirical net disbursements distributions, the results are not entirely trivial. The equation for the distribution is:

\[(A5) \quad f(D) = 1/(P - R),\]

where \(P\) and \(R\) represent the payment and receipt limits, respectively. Since \(R\) is assumed equal to \(-P\),

\[(A6) \quad f(D) = 1/2P.\]

Substituting this expression into equation \((A4)\) and solving for \(M\) produces the following equation for optimal precautionary cash balances:

\[M = \frac{P(v - 2r) + c}{v}.\]

This equation shows that precautionary cash balances optimally will vary linearly and positively with \(P\) and \(c\), inversely with \(r\), and positively and curvilinearly with \(v\), subject to the condition that:

\[v \cdot P > 2 \cdot r \cdot P > c.\]

The triangular distribution, which intuitively appears to be a better approximation of empirical net disbursements distributions, takes the form:

\[(A8) \quad f(D) = 2(1/P - D/P^2). \quad 0 \leq D \leq P.\]

Substituting this expression into equation \((A4)\) and solving for \(M\) produces the following equation for optimal precautionary cash balances:

\[M = P + \frac{c - \sqrt{c^2 + r \cdot v \cdot P^2}}{v}.\]

This equation similarly suggests that optimal precautionary cash balances will vary curvilinearly and positively with \(P\), \(c\), \(v\), and the reciprocal of the opportunity cost rate, \(r\).